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A TREATISE

ON

DYNAMICS OF A PARTICLE.

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A TREATISE ON

DYNAMICS OF A PARTICLE,

WITH NUMEROUS EXAMPLES.

BY

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PREFACE.

THIS work, commenced by Mr STEELE and myself towards the end of 1852, first appeared in 1856. At Mr STEELE'S early death his allotted share of the work was uncompleted, and I had to undertake the final arrangement of the whole. In the subsequent editions it has derived much benefit from revision: first by Mr STIRLING of Trinity in 1865, then by Mr W. D. NIVEN of Trinity in 1871, and recently by Prof. GREENHILL of Emmanuel in 1878.

It now appears after a general revision by myself, with the assistance of Dr C. G. KNOTT and of my colleague Prof. CHRYSTAL.

Under such circumstances it could not fail to be a patchwork of a somewhat complicated kind; but the comparatively rapid exhaustion of the latest edition shows that, with all its many faults, it meets not very inadequately a real want.

I have no doubt that, with a few months' leisure, I could immensely improve it; if merely by giving it more unity of plan. But the time I am able to devote to such things has to be snatched at irregular intervals from other engrossing work; and I am led, therefore, very naturally rather to the making of hastily improvised insertions than to carrying out any well-considered scheme of compression or co-ordination.

The book's most important fault is its bulk; yet I do not think it can be honestly accused of prolixity. And I have always considered undue prolixity to be, next of course to inaccuracy, the greatest fault that a scientific work could exhibit. The number of Examples is perhaps unduly large, but experience has shown me that there are many readers who will not consider this a defect.

My attention has been called to the fact that several sections of this book, in which some novelties appear, have been translated almost *letter for letter* and transferred, without the slightest allusion to their source, to the pages of a German work. Several other books have obviously been similarly treated by the same compiler. It is well that this should be generally known, as the British authors might otherwise come to be supposed to have adopted these passages *simpliciter* from the German.

P. GUTHRIE TAIT.

College, Edinburgh,
July, 1882.

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DYNAMICS OF A PARTICLE.

CHAPTER I.

KINEMATICS.

1. *Dynamics* is the Science which investigates the action of Force; and naturally divides itself into two parts as follows.

2. Force is recognized as acting in two ways: in *Statics* so as to compel rest or to prevent change of motion, and in *Kinetics* so as to produce or to change motion.

3. In Kinetics it is not mere *motion* which is investigated, but the relation of *forces* to motion. The circumstances of mere motion, considered without reference to the bodies moved, or to the forces producing the motion, or to the forces called into action by the motion, constitute the subject of a branch of Pure Mathematics, which is called *Kinematics*. To this, as a necessary introduction, we devote the present chapter.

4. The rate of motion (or the rate of change of *position*) of a point is called its *Velocity*. It is greater or less as the space passed over in a given time is greater or less: and it may be *constant*, *i.e.* the same at every instant; or it may be *variable*.

Constant velocity is measured by the space passed over in unit of time, and is, in general, expressed in feet per second; if very great, as in the case of light, it may be measured in miles per second. It is to be observed, that *Time* is here used in the abstract sense of a uniformly-increasing quantity

—what in the differential calculus is called an independent variable. Its physical definition is given in Chap. II.

5. Thus, a point moving uniformly with the velocity v describes a space of v feet each second, and therefore vt feet in t seconds, t being any number whatever. Putting s for the space described in t seconds, we have

$$s = vt.$$

Hence with unit velocity a point describes unit of space in unit of time. The path may be straight or curved.

6. It is well to observe that since, by our formula, we have generally

$$v = \frac{s}{t},$$

and since nothing has been said as to the magnitudes of s and t, we may take these as small as we choose. Thus *we get the same result whether we derive v from the space described in a million seconds, or from that described in a millionth of a second.* This idea is very useful, as it will give confidence in results when a variable velocity has to be measured, and we find ourselves obliged to approximate to its value by considering the space described in an interval so short, that during its lapse the velocity does not sensibly alter in value.

7. Velocity is said to be variable when the moving point does not describe equal spaces in equal times. *The velocity at any instant is then measured by the space which would have been described in a unit of time, if the point had moved on uniformly for that interval with the velocity which it had at the instant contemplated.* This is a most important, and in fact a fundamental, conception, which the student must thoroughly realize before he can usefully proceed farther. It lies at the root of all the correct methods ever devised for the purpose of measuring the rate at which *change*, of any kind, is going on.

Let v be the velocity of the point at the time t, measured from a fixed epoch, s the space described by it during that time, and $s + \delta s$ the space described during a greater interval

$t+\delta t$. Suppose v_1 to be the greatest, and v_2 the least, velocity with which the point moves during the time δt; then $v_1\delta t$, $v_2\delta t$ would be the spaces which a point would describe in that interval, moving uniformly with these velocities respectively. But the actual velocity of the point is not greater than v_1, and not less than v_2, therefore as regards the actual space described,

δs is not greater than $v_1\delta t$, and not less than $v_2\delta t$,

or $\dfrac{\delta s}{\delta t}$ ....v_1........................ v_2,

however small δt may be. But, as δt continually diminishes, v_1 and v_2 tend continually to, and ultimately become each equal to, v. Therefore, proceeding to the limit,

$$\frac{ds}{dt}=v.$$

If v be negative in this expression, it indicates that s diminishes as t increases; the positive case, which we have taken as the standard one, referring to that in which s and t increase together. It follows that, if a velocity in one direction be considered positive, in the opposite direction it must be considered negative; and consequently the sign of the velocity indicates the direction of motion, when the path is given.

This investigation rests on the supposition that the velocity alters continuously, and not by jerks. It would require an infinite force to produce in an infinitely short time such a change of velocity in a *material* particle. Hence as we are preparing for physical applications only, such cases may be excluded for the present. The action of great forces for short periods of time will be treated in the chapter on *Impact*.

8. So far as we have yet spoken of it, velocity has been regarded merely as *speed*, and all that is said above is equally applicable whether the point be considered as moving in a straight, or in a curved, line. In the latter case, however, the direction of motion continually changes; and it is necessary to know at every instant the direction, as well as the magnitude, of the point's velocity. This is usually, and in

general most conveniently, done by considering the velocities of the point parallel to the three co-ordinate axes respectively. In fact velocity is properly a *directed magnitude* (or *vector*, as it is now called) involving at once the direction and the speed of the motion. If the co-ordinates of the moving point be represented by x, y, z, the rates of increase of these, or the velocities parallel to the corresponding axes, will by reasoning analogous to that in § 7 be

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}.$$

Denoting by v the *speed* of the motion, we have

$$v = \frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}};$$

and, if α, β, γ be the angles which the *direction* of the motion makes with the axes,

$$\cos\alpha = \frac{dx}{ds} = \frac{\frac{dx}{dt}}{\frac{ds}{dt}};$$

or $$\frac{dx}{dt} = v\cos\alpha = v_x, \text{ suppose.}$$

Similarly, $$\frac{dy}{dt} = v\cos\beta = v_y,$$

$$\frac{dz}{dt} = v\cos\gamma = v_z.$$

Hence, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are to be found from the whole velocity v, by *resolving* as it is called; *i.e.* by multiplying by the direction-cosines of the direction of motion. They are called the *Component Velocities* of the point: and, with reference to them, v is called the *Resultant Velocity*.

9. It follows from the above, that, if a point be moving in any direction, we may suppose its velocity to be the resultant of three coexistent velocities in any three directions at

right angles to each other; or, more generally, in any three directions not coplanar. But the rectangular resolution is the simplest and best except in some very special applications.

Let v_x, v_y, v_z be the rectangular components of the velocity v of a moving point, then the resolved part of v along a line inclined at angles λ, μ, ν to the axes will be

$$v_x \cos\lambda + v_y \cos\mu + v_z \cos\nu.$$

For, let α, β, γ be the angles which the direction of the point's motion makes with the axes, θ the angle between this direction and the given line. Then since

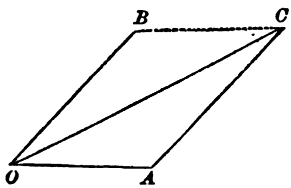
$$\cos\theta = \cos\alpha . \cos\lambda + \cos\beta \cos\mu + \cos\gamma \cos\nu$$

the resolved part of v along that line is

$$v\cos\theta = v\{\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu\}$$
$$= v_x \cos\lambda + v_y \cos\mu + v_z \cos\nu.$$

10. These propositions are virtually equivalent to the following obvious geometrical construction, which is the Law of Composition of Vectors :—

To compound any two velocities as OA, OB in the figure; where OA, for instance, represents in magnitude and direction the space which would be described in one second by a point moving with the first of the given velocities—and



similarly OB for the second; from A draw AC parallel and equal to OB. Join OC:—then OC is the resultant velocity in magnitude and direction. For the motions parallel to OA and OB are independent.

OC is evidently the diagonal of the parallelogram two of whose sides are OA, OB.

Hence the resultant of any two velocities as OA, AC, in the figure is a velocity represented by the third side, OC, of the triangle OAC.

Hence if a point have, simultaneously, velocities represented by OA, AC, and CO, the sides of a triangle *taken in the same order*, it is at rest.

Hence the resultant of velocities represented by the sides of any closed polygon whatever, whether in one plane or not, taken all in the same order, is zero.

Hence also the resultant of velocities represented by all the sides of a polygon but one, taken in order, is represented by that one taken in the opposite direction.

When there are two velocities or three velocities in two or in three rectangular directions, the resultant is the square root of the sum of their squares—and the cosines of the inclination of its direction to the given directions are the ratios of the components to the resultant.

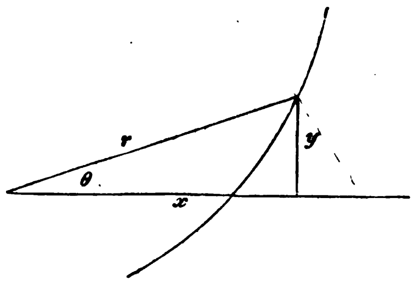
[Newton's Method of *Fluxions* was devised simply to express this and other fundamental conceptions in Kinematics. To him $\dot{s}, \dot{x}, \dot{y}, \dot{z}$, or (as we now somewhat less conveniently write them) $\frac{ds}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, are simply the velocity of the moving point and its components parallel to the axes. It may be convenient, or even necessary, to use the idea of *Limits* or of *Infinitesimals* to calculate their values; but the Fluxions themselves do not involve any such idea.]

11. *When a point moves in a plane curve, to express its component velocities at any instant along, and perpendicular to, the radius vector drawn from a fixed point in the plane of the curve.*

Let x, y be the rectangular, r, θ the polar, co-ordinates of the moving point; so that

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We have at once, by differentiation,



$$\left.\begin{aligned}\frac{dx}{dt}&=\frac{dr}{dt}\cos\theta-r\sin\theta\frac{d\theta}{dt}\\ \text{and}\qquad \frac{dy}{dt}&=\frac{dr}{dt}\sin\theta+r\cos\theta\frac{d\theta}{dt}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(1),$$

which are the velocities parallel to x and y. But by § 9 the velocity along the radius vector is

$$\frac{dy}{dt}\sin\theta+\frac{dx}{dt}\cos\theta=\frac{dr}{dt}, \text{ by (1)};$$

and the velocity perpendicular to it is

$$\frac{dy}{dt}\cos\theta-\frac{dx}{dt}\sin\theta=r\frac{d\theta}{dt}, \text{ by (1)}.$$

12. The velocity of a point (in the sense of its *speed*) is popularly said to be accelerated or retarded according as it increases or diminishes, but the word *Acceleration* is scientifically used in both senses; and may be defined as the rate of change of the velocity per unit of time.

Acceleration may be either constant or variable. It is said to be constant when the point receives equal increments of velocity in equal times, and is then measured by the actual increase of velocity generated in unit of time. Let the unit of acceleration be so taken that a point under its action would receive an increment of a unit of velocity in a unit of time;

then a point under the influence of α units of acceleration would receive an increment of α units of velocity in a unit of time, and consequently αt units of acceleration in t units of time. If the point starts from rest we have

$$v = \alpha t,$$

where v denotes the velocity at the end of the interval t, and α the acceleration.

13. Acceleration is variable when the point does not receive equal increments of velocity in equal increments of time. The acceleration at any instant is then measured by the increment of velocity which would have been generated in a unit of time had the acceleration remained constant during that interval and equal to the value at its commencement.

Let v be the velocity of the point at the end of the time t, α the acceleration at that instant, $v + \delta v$ the velocity at the end of the time $t + \delta t$; and let α_1, α_2 be the greatest and least values of the acceleration during the interval δt, then $\alpha_1 \delta t$, $\alpha_2 dt$ would be the increments of velocity in that interval, of a point under those accelerations respectively. But the actual acceleration is not greater than α_1 and not less than α_2, therefore the actual increment of velocity

δv is not greater than $\alpha_1 \delta t$ and not less than $\alpha_2 \delta t$,

or $\dfrac{\delta v}{\delta t}$ .... α_1 .... α_2,

however small δt may be. But, as δt continually diminishes, α_1 and α_2 tend continually to and ultimately become each equal to α. Therefore, proceeding to the limit,

$$\frac{dv}{dt} = \alpha$$

The positive sign given to α shews that v increases with t, while a negative sign would shew that v decreases as t increases, in other words a negative acceleration is a retardation.

Combining the above equation with

$$\frac{ds}{dt} = v,$$

we have

$$\frac{d^2s}{dt^2} = a,$$

considering t as the independent variable.

[Here, again, Newton employs the symbol $\ddot{s}$ to represent the rate of increase of $\dot{s}$, a quantity whose *conception* is altogether independent of the methods (infinitesimal or not) which may be employed to calculate its value.]

14. Thus far we have been dealing with a point's motion in some *definite* path, which may be either straight or curved, but in which there is only *one* degree of freedom to move, and in which therefore the position at any time is determined by *one* variable, s. But when we consider velocity as a directed magnitude we are led to generalize the definition of Acceleration (see § 20 below).

If the path be curved, the accelerations of the rates of increase of the co-ordinates of the moving point are called the *Component Accelerations* parallel to the axes. If these be denoted by a_x, a_y, a_z, we shall have

$$\frac{d^2x}{dt^2} = a_x, \quad \frac{d^2y}{dt^2} = a_y, \quad \frac{d^2z}{dt^2} = a_z.$$

With reference to these, $\sqrt{a_x^2 + a_y^2 + a_z^2}$ is called the *Resultant Acceleration.*

15. The acceleration $\frac{d^2s}{dt^2}$ is not the complete resultant of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, as may easily be seen: for its square does not in general equal the sum of the squares of those three accelerations. It is, however, the only part of their resultant which has any effect on the *magnitude* of the velocity; in short $\frac{d^2s}{dt^2}$ is the sum of the resolved parts of

$\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ in the direction of motion, as the following identical equation shews:

$$\frac{d^2s}{dt^2} = \frac{dx}{ds}\frac{d^2x}{dt^2} + \frac{dy}{ds}\frac{d^2y}{dt^2} + \frac{dz}{ds}\frac{d^2z}{dt^2}.$$

This follows immediately from the equation of § (8)

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

by differentiation. And it shews that acceleration is to be resolved according to the same law as velocity. For to find $\frac{d^2s}{dt^2}$, the acceleration along s, $\frac{d^2x}{dt^2}$ has to be multiplied by $\frac{dx}{ds}$, &c. &c. which is the vector law.

The other part of the resultant is at right angles to this, and its sole effect is to change the direction of the motion of the point. And this leads us to another form of acceleration, viz. when the magnitude of the velocity is unaltered, but the *direction* of motion changes. Its value in terms of the velocity and the curvature will be given later.

The above equation also shews, since $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are the direction-cosines of the small arc ds which may have any direction whatever, that to obtain the acceleration along any line inclined at given angles to the axes, we must resolve the component accelerations parallel to the axes along it, and take the sum of the resolved parts. Thus the acceleration along a line inclined at angles λ, μ, ν to the axes is

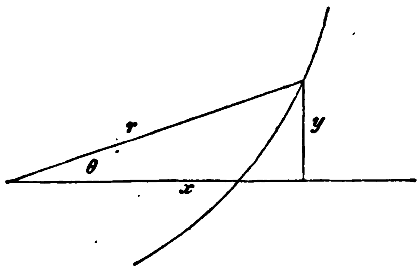
$$\alpha_x \cos\lambda + \alpha_y \cos\mu + \alpha_z \cos\nu.$$

16. *A point moves in a plane curve, to express its component accelerations at any instant along, and perpendicular to, the radius vector.*

Let x, y be the rectangular, r, θ the polar, co-ordinates; so that

$$x = r\cos\theta,$$
$$y = r\sin\theta;$$

we have $$\frac{dx}{dt}=\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt},$$



and $$\frac{d^2x}{dt^2}=\left\{\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right\}\cos\theta-\left(2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}\right)\sin\theta.$$

Similarly,

$$\frac{d^2y}{dt^2}=\left\{\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right\}\sin\theta+\left(2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}\right)\cos\theta.$$

These are the accelerations parallel to x and y. And since, by § 15, the acceleration along the radius vector is

$$\frac{d^2y}{dt^2}\sin\theta+\frac{d^2x}{dt^2}\cos\theta,$$

the above expressions give it in the form

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2.$$

The acceleration perpendicular to the radius vector is

$$\frac{d^2y}{dt^2}\cos\theta-\frac{d^2x}{dt^2}\sin\theta,$$

that is, $$2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2},$$

which may be written $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$.

17. *When a point is in motion in any curve, to find its accelerations along, and perpendicular to, the tangent, at any instant.*

Let x, y, z be the co-ordinates of the point at the end of the time t, s the length of the arc described during that interval. Then, since by the equations of the curve x, y and z are functions of s,

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt};$$

and $$\frac{d^2x}{dt^2} = \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dx}{ds}\frac{d^2s}{dt^2}.$$

Similarly, $$\frac{d^2y}{dt^2} = \frac{d^2y}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dy}{ds}\frac{d^2s}{dt^2},$$

$$\frac{d^2z}{dt^2} = \frac{d^2z}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dz}{ds}\frac{d^2s}{dt^2}.$$

Remembering the law of resolution of acceleration, the *form* of these equations shews that in them are resolved along x, y, z, 1st an acceleration $\frac{d^2s}{dt^2}$, whose direction-cosines are $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, and 2nd an acceleration $\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$, whose direction-cosines are $\rho\frac{d^2x}{ds^2}$, $\rho\frac{d^2y}{ds^2}$, $\rho\frac{d^2z}{ds^2}$; where ρ is a linear quantity, which will be presently recognized as the radius of curvature of the path. This process might have been employed with advantage in some previous sections. But, for the beginner, we must take a more laborious method.

18. To find the acceleration along the tangent, we must multiply these component accelerations by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, respectively, and add. Thus the tangential acceleration is

$$\frac{dx}{ds}\frac{d^2x}{dt^2} + \frac{dy}{ds}\frac{d^2y}{dt^2} + \frac{dz}{ds}\frac{d^2z}{dt^2} = \frac{d^2s}{dt^2} = \frac{dv}{dt},$$

as we have already seen. Also in the normal, towards the centre of curvature, we have the acceleration

$$\rho\left(\frac{d^2x}{ds^2}\frac{d^2x}{dt^2}+\ldots\ldots\right)=\rho\left(\frac{ds}{dt}\right)^2\left\{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right\}$$

$$=\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2=\frac{v^2}{\rho}.$$

We assume here the following equations from Analytical Geometry,

739,62

$$\frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2,$$

where ρ is the radius of curvature, whose direction-cosines are

$$\rho\frac{d^2x}{ds^2},\ \rho\frac{d^2y}{ds^2},\ \rho\frac{d^2z}{ds^2};$$

and $$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1,$$

whence $$\frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\frac{d^2y}{ds^2}+\frac{dz}{ds}\frac{d^2z}{ds^2}=0^*.$$

---

* The accelerations of the moving point may be found in the following manner. There is obviously no acceleration perpendicular to the osculating plane, as that plane contains two consecutive directions of the point's motion. Of the two consecutive directions let the first make an angle θ with any fixed line in the osculating plane, then $v\cos\theta$ and $v\sin\theta$ are the velocities of the point parallel and perpendicular to the fixed line respectively. Consequently $\frac{d}{dt}(v\cos\theta)$ and $\frac{d}{dt}(v\sin\theta)$ are the accelerations in the same directions. These expressions, when expanded, become $\frac{dv}{dt}\cos\theta-v\sin\theta\frac{d\theta}{dt}$, and $\frac{dv}{dt}\sin\theta+v\cos\theta\frac{d\theta}{dt}$. Therefore the accelerations along the tangent and the normal are $\frac{dv}{dt}$ and $v\frac{d\theta}{dt}$, the last being positive in the direction of the centre of curvature. Since $\frac{d\theta}{ds}=\frac{1}{\rho}$, the normal acceleration, being $=v\frac{d\theta}{ds}\cdot\frac{ds}{dt}$, may be expressed as $\frac{v^2}{\rho}$.

19. We might have treated the component accelerations thus

$$\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2 \text{ or (resultant acceleration)}^2$$

$$= \frac{1}{\rho^2}\left(\frac{ds}{dt}\right)^4 + \left(\frac{d^2s}{dt^2}\right)^2,$$

by adding the squares of their values as given in § 17.

Now $\frac{d^2s}{dt^2}$ is the acceleration along the tangent, and the other part $\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$, or $\frac{v^2}{\rho}$, acts at right angles to it as the form of the equation shews, and consequently is the acceleration perpendicular to the tangent.

From the expressions for $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$, we also obtain

$$\frac{d^2x}{dt^2}\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right)$$

$$+\frac{d^2y}{dt^2}\left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right)$$

$$+\frac{d^2z}{dt^2}\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right) = 0;$$

which may be written in the form of a determinant

$$\begin{vmatrix} \frac{d^2x}{dt^2} & \frac{d^2y}{dt^2} & \frac{d^2z}{dt^2} \\ \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \end{vmatrix} = 0.$$

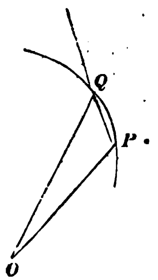
This signifies that the Resultant Acceleration lies in the plane containing the tangent and the radius of absolute curvature, or that there is no acceleration perpendicular to the

osculating plane. The acceleration $\frac{v^2}{\rho}$ must therefore be along a normal to the path drawn in the osculating plane; that is, along the radius of absolute curvature.

20. We are therefore led to *expand* the definition given in § 12 thus:—Acceleration is the *rate of change of velocity whether that change take place in the direction of motion or not.*

What is meant by change of velocity is evident from § 10. For if a velocity OA (in the figure of that section) become OC, its change is AC, or OB.

Hence, just as the direction of motion of a point is the tangent to its path—so the direction of acceleration of a moving point is to be found by the following construction.

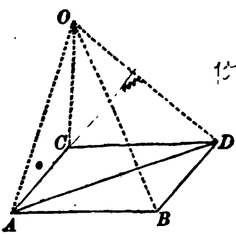


From any point O draw lines OP, OQ, etc., representing in magnitude and direction the velocity of the moving point at every instant. The points, P, Q, etc., form in all cases of motion of a material particle a continuous curve, for an infinitely great force is requisite to change the velocity of a particle *abruptly* either in direction or magnitude. Now if Q be a point near to P, OP and OQ represent two successive values of the velocity. Hence PQ is the whole change of velocity during the interval. As the interval becomes smaller, the direction PQ more and more nearly becomes the tangent at P. Hence the direction of acceleration is that of the tangent to the curve thus described, called by its inventor, Sir W. R. Hamilton, the *Hodograph.*

The amount of acceleration is the rate of change of velocity, and is therefore measured by the velocity of P in the curve PQ.

21. The *Moment* of a velocity about any point is the rectangle under its magnitude and the perpendicular from the point upon its direction. *The moment of the resultant velocity of a point about any point in the plane of the components is equal to the algebraic sum of the moments of the components, the proper sign of each moment depending on the direction of motion about the point.* The same is true of moments of acceleration, and of moments of momentum as defined later.

Consider two component velocities, AB and AC, and let AD be their resultant (§ 10). Their half moments round



the point O are respectively the areas OAB, OCA. Now OCA, together with half the area of the parallelogram $CABD$, is equal to OBD. Hence the sum of the two half moments together with half the area of the parallelogram is equal to AOB together with BOD, that is to say, to the area of the whole figure $OABD$. But ABD, a part of this figure, is equal to half the area of the parallelogram; and therefore the remainder, OAD, is equal to the sum of the two half moments. And OAD is half the moment of the resultant velocity round the point O. Hence the moment of the resultant is equal to the sum of the moments of the two components. By attending to the *signs* of the moments, we see that the proposition holds when O is within the angle CAB.

22. Now if the direction of one of the components always passes through the point O, its moment vanishes. This is the case of a motion in which the acceleration is directed to a

fixed point, and we thus prove the theorem that *in the case of acceleration always directed to a fixed point the path is plane and the areas described by the radius-vector are proportional to the times;* for the moment of velocity, which in this case is constant, is evidently double the rate at which the area is traced out by the radius-vector.

23. Hence in this case the velocity at any point is inversely as the perpendicular from the fixed point upon the tangent to the path, the momentary direction of motion.

For evidently the product of this perpendicular and the velocity at any instant gives double the area described in one second about the fixed point, which has just been shewn to be a constant quantity.

24. The results of the last three sections may be easily obtained analytically, thus. Let the plane of motion be taken as that of x, y; and let the origin be the point about which moments are taken. Then if x, y be the position of the moving point at time t, the perpendicular from the origin on the tangent to its path is

$$p = x\frac{dy}{ds} - y\frac{dx}{ds} = r^2\frac{d\theta}{ds}, \text{ in polar co-ordinates.}$$

From this we have at once

$$p\frac{ds}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

or with the notation of § 8,

$$pv = xv_y - yv_x,$$

which is the theorem of § 19.

Also $$\frac{d}{dt}(pv) = x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

accel.

Now, if the acceleration be directed to or from O, its moment about O, which is evidently

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2},$$

must vanish. Hence (2) gives

$$pv = \text{constant, which is § 23.}$$

By means of (1) this gives

$$r^2\frac{d\theta}{dt} = \text{constant, which is § 22;}$$

since, if A be the area traced out by the radius-vector,

$$\frac{dA}{d\theta} = \frac{r^2}{2}.$$

25. *To determine the motion of a point when the acceleration of its velocity is given.*

This is one of the most general of the Problems suggested by the Kinematics of a point, for it includes, as will be seen, the determination of the motion when the component velocities are given.

Let α, β, γ be the components of the given acceleration; we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= \alpha,\\ \frac{d^2y}{dt^2} &= \beta,\\ \frac{d^2z}{dt^2} &= \gamma,\end{aligned}\right\}\quad\text{..................(1)}$$

Now α, β, γ may be functions of x, y, z, t, $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, or $\dfrac{dz}{dt}$, or of two or more of these quantities. Equations (1) must be integrated as simultaneous differential equations if possible.

Thus by one integration we have the values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, in terms of one or more of the quantities x, y, z and t; that is, the component velocities are known.

Another integration, if it can be performed, gives x, y, and z, in terms of t; and, if the latter variable be eliminated from the three integrated equations, we have the two equations of the path in space: and thus, theoretically at least, the motion is completely determined.

It is unnecessary to give examples of the integration of such equations here, as the major part of the following chapters will be devoted to them.

26. So far for a single point. When more points than one are considered, Kinematics enables us to determine, from the given motions of all, their *relative* motions with respect to any one of them; or conversely, from the actual motion of one, and the motions relative to it of the others, to determine the *actual* motions of the latter in space. This depends on the following self-evident proposition.

If the velocity of any point of a system be reversed in direction, and be communicated to each point of the system in composition with that which it already possesses, the relative motions of all about the first, thus reduced to rest, will be the same as their relative motions about it when all were in motion.

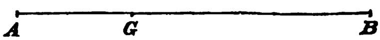
For the proof it is sufficient to notice that if at every instant the distance of two points, and the direction of the line joining them be the same as for two other points, the relative motions of one of each pair about the other will be the same. The simplest illustrations of this proposition are furnished by the relative motions of objects in a vessel or carriage, which are independent of the common velocity of the whole—or, on a grander scale, of terrestrial objects, whose relative motions are unaffected by the earth's rotation, or by its motion in space.

Since accelerations are compounded according to the same law as velocities, the above theorem is true of them also.

27. *Two points describe similar orbits about each other and about any point dividing in a given ratio the line which joins them.*

Let A and B be the points, G a point in AB such that $\frac{AG}{GB} =$ a constant.

The path of B about A will evidently be the same as that of A about B, since the length and direction of the



line AB are the same whichever end be supposed fixed. Also if G be fixed, the path of B about it will evidently differ from that of B about A by having corresponding radii-vectores diminished in the ratio $\frac{BG}{AB}$. But this is the definition of similar curves. The same of course would hold with respect to the relative path of A with respect to G. This proposition will be found of considerable use afterwards, as it enables us materially to simplify the equations of motion of two mutually attracting free particles.

28. As an instance of relative motion, *consider two points, one of which moves uniformly in a straight line, while the other moves uniformly in a circle about the first as centre; determine the path of the second point, the motion being in one plane.*

Take the line of motion of the first as the axis of x, v its velocity, the plane of the circle as that of xy, a the radius of the relative circular orbit, ω the angular velocity in it, § 37. Suppose the revolving point to be initially in the axis. Also at time t suppose the line joining the points to be inclined at an angle θ to the axis of x. Then for the co-ordinates of the revolving point we have

$$y = a \sin \theta,$$
$$x = vt + a \cos \theta.$$

But $$\theta = \omega t;$$

hence $$x = \frac{v}{\omega}\sin^{-1}\frac{y}{a} + \sqrt{(a^2 - y^2)}$$

is the equation of the absolute path required. This belongs to the class of cycloids; it is prolate or curtate according as v is greater or less than $a\omega$, or the absolute motion of the first point greater or less than that of the other in its circular orbit. If the two are equal, or $v = a\omega$, we have the equation of the common cycloid, as is indeed evident, for the circular path may be supposed the generating circle, and the velocity of the centre in its rectilinear path is equal to that of the tracing point about that centre.

29. It is evident that, whatever be the relative path, if r, θ denote the relative co-ordinates of the second point with respect to the first at time t, x, y, and $\bar{x}$ the absolute co-ordinates at the same time,

$$\left.\begin{array}{l} x = \bar{x} + r\cos\theta \\ y = r\sin\theta \end{array}\right\} \text{......} (1).$$

Now in the first case, when the motion of the first point, and that in the relative orbit are given, $\bar{x}$, r, and θ are known functions of t; if therefore these values be substituted in (1), and t be eliminated, we shall have the equation between x and y, which is required.

Again, if the absolute orbits of both are given, x, y, and $\bar{x}$ are given in terms of t, and thus equations (1) serve to give r and θ in terms of t, which furnishes the complete determination of the relative path, and the circumstances of its description.

30. The following is a most useful case, having many important applications in Physical Optics, &c.

A point A *is fixed.* B *describes uniformly a circle about* A, *and* C *describes uniformly (in the same plane) a circle about* B. *Find the motion of* C *relative to* A.

Let a be the length of AB, b that of BC, r that of AC; and at time t let them make angles ϕ, χ, θ with some fixed line in the plane of motion. Then

$$r\cos\theta = a\cos\phi + b\cos\chi,$$
$$r\sin\theta = a\sin\phi + b\sin\chi.$$

But ϕ and χ increase uniformly. Hence

$$\phi = mt + \alpha,$$
$$\chi = nt + \beta,$$

where m, n, α, β, are constants. Thus

$$r\cos\theta = a\cos(mt+\alpha) + b\cos(nt+\beta),$$
$$r\sin\theta = a\sin(mt+\alpha) + b\sin(nt+\beta).$$

These are the general equations of Epicycloids and Hypocycloids; and from them all their properties may be derived.

We confine ourselves to one or two very simple cases.

(1) Let $m = n$, $a = b$. (This is the composition of two equal circular motions, in the same direction and of equal period.) We have

$$r\cos\theta = 2a\cos\frac{\alpha-\beta}{2}\cos\left(mt+\frac{\alpha+\beta}{2}\right),$$
$$r\sin\theta = 2a\cos\frac{\alpha-\beta}{2}\sin\left(mt+\frac{\alpha+\beta}{2}\right);$$

whence

$$r = 2a\cos\frac{\alpha-\beta}{2}$$
$$\theta = mt + \frac{\alpha+\beta}{2}.$$

This also denotes uniform circular motion, and of the same period, and in the same direction, as the components.

(2) Let $m = -n$, $a = b$. (Here we compound equal circular motions, of equal period, but in opposite directions.) As before we have

$$r\cos\theta = 2a\cos\frac{\alpha+\beta}{2}\cos\left(mt+\frac{\alpha-\beta}{2}\right),$$
$$r\sin\theta = 2a\sin\frac{\alpha+\beta}{2}\cos\left(mt+\frac{\alpha-\beta}{2}\right).$$

Therefore

$$r = 2a \cos\left(mt + \frac{\alpha - \beta}{2}\right),$$

$$\theta = \frac{\alpha + \beta}{2};$$

and this denotes vibratory motion in a definite straight line.

31. *In any system of moving points, to determine the relative from the absolute motions; and vice versâ.*

Let $x_1, y_1, z_1, x_2, y_2, z_2$ be the co-ordinates of two of the points, x, y, z the relative co-ordinates of the second with regard to the first, $u_1, v_1, w_1, u_2, v_2, w_2$ the velocities of each parallel to the axes, u, v, w the velocities of the second relatively to the first.

Then

$$\begin{aligned} x &= x_2 - x_1, & u &= u_2 - u_1, \\ y &= y_2 - y_1, & v &= v_2 - v_1, \\ z &= z_2 - z_1, & w &= w_2 - w_1. \end{aligned}$$

The second group may be derived from the first by differentiation with respect to t.

Now, when the *actual* motions of the two are given, all the subscribed quantities are known. Hence the above equations give the circumstances of the relative motion.

Or if the actual motion of the first, and the relative motion about it of the second, be known, we have $x\,y\,z, u\,v\,w, x_1\,y_1\,z_1, u_1\,v_1\,w_1$, to find the other six quantities for the actual motion of the second in space.

A second differentiation proves the statement in § 26 regarding relative acceleration.

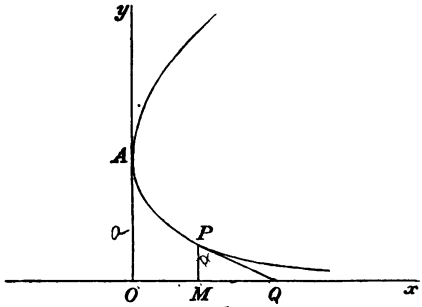
32. Some of the best illustrations of this part of our subject are to be found in what are called *Curves of Pursuit.*

These questions arose from the consideration of the path taken by a dog, who in following his master always directs his course towards him.

In order to simplify the question the rates of motion of both master and dog are supposed to continue constant; or at least to have a constant ratio.

33. As an instance of the curve of pursuit, suppose it be required *to determine the path of a point* P *which continually, with constant velocity* u, *moves towards another point* Q *which is describing a straight line with constant velocity* v.

The curve of course is plane. Take the line of motion of the second point Q as the axis of x, and let $\bar{x}$ denote its position at the instant when the co-ordinates of the first,



P, are x, y. The axis of y is chosen as that tangent to the curve of pursuit which is perpendicular to the axis of x, and the distance between the points in that position is a.

Let $\frac{v}{u} = e$, then by the conditions of the problem we have

$$eAP = OQ,$$

and PQ a tangent at P.

Expressed analytically these lead to the following equations;

$$es = \bar{x} = x - y\frac{dx}{dy}.$$

The mode of solution is precisely the same whether x or y be taken as independent variable: but y is to be preferred as it leads to less cumbrous expressions.

Differentiating therefore with respect to y, we have

$$e\frac{ds}{dy} = -y\frac{d^2x}{dy^2}.$$

But s increases as y diminishes,

whence $$\frac{ds}{dy} = -\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}.$$

Hence $$\frac{e}{y} = \frac{\frac{d^2x}{dy^2}}{\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}}.$$

Integrating, and noting that $y = a$, $\frac{dx}{dy} = 0$, together,

$$e \log \frac{y}{a} = \log\left[\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} + \frac{dx}{dy}\right].$$

Hence, $$\left(\frac{y}{a}\right)^e = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} + \frac{dx}{dy},$$

and therefore, taking reciprocals,

$$\left(\frac{a}{y}\right)^e = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} - \frac{dx}{dy}.$$

Subtracting, we have finally

$$2\frac{dx}{dy} = \left(\frac{y}{a}\right)^e - \left(\frac{a}{y}\right)^e \quad \text{..........................(1)},$$

$$\text{or } 2(x + C) = \frac{y^{e+1}}{a^e(e+1)} + \frac{a^e}{y^{e-1}(e-1)}.$$

But $x = 0$, $y = a$, together; which gives $C = \frac{ae}{e^2 - 1}$.

$$\text{Hence } 2\left(x + \frac{ae}{e^2 - 1}\right) = \frac{y^{e+1}}{a^e(e+1)} + \frac{a^e}{y^{e-1}(e-1)} \quad \text{........(2)}.$$

This is the correct integral for all values of e except unity, when it ceases to have any meaning. To this case we will presently recur.

There are two cases of curves represented by equation (2), 1st, $e > 1$, 2nd, $e < 1$.

In the first case Q moves the faster, and P can never overtake it; the curve therefore never meets the axis of x, which indeed will be seen by (2) to be an asymptote.

In the second case equation (2) becomes

$$2\left(x - \frac{ae}{1-e^2}\right) = \frac{y^{1+e}}{a^e(1+e)} - \frac{a^e y^{1-e}}{1-e},$$

and for $x = \frac{ae}{1-e^2}$ we have $y=0$, and also by (1) $\frac{dx}{dy}$ infinite. Hence the curve touches the axis at this point. The remainder of the curve satisfies an obvious modification of the question, whence it is called the *Curve of Flight*. $\Bigl\{$It is to be observed, however, that $x = \frac{ae}{1-e^2}$ gives also $y = \pm a\left(\frac{1+e}{1-e}\right)^{\frac{1}{2e}}\Bigr\}$.

The distance between P and Q, being

$$\sqrt{(\bar{x}-x)^2 + y^2},$$

is easily seen by the fundamental equations to be

$$\pm y\frac{ds}{dy},$$

or, by (1),

$$\pm\frac{y}{2}\left\{\left(\frac{y}{a}\right)^e + \left(\frac{a}{y}\right)^e\right\};$$

where the sign is to be chosen so as to make the expression positive.

When $e > 1$, this expression is infinite both for $y = \infty$ and for $y = 0$. The minimum value is easily found to be

$$\frac{ae}{\sqrt{e^2-1}}\left(\frac{e-1}{e+1}\right)^{\frac{1}{2e}}.$$

When $e < 1$, the distance vanishes, as we have seen it must, when $y = 0$.

34. When $e = 1$, the corrected integral of (1) is

$$2\left(x + \frac{a}{4}\right) = \frac{y^2}{2a} - a\log\frac{y}{a}.$$

This is the only case in which we do not obtain an algebraic curve. Here again the axis of x is an asymptote, and we easily find

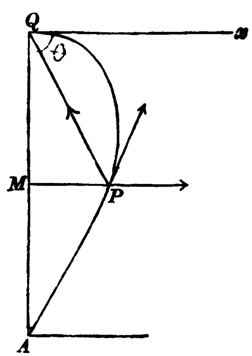
$$PQ = \frac{y^2}{2a} + \frac{a}{2},$$

which shews that the limit to which the distance tends is $\frac{a}{2}$. The same result may at once be obtained by putting $e = 1$ in the expression for the minimum distance found above in the case of $e > 1$.

35. As an instance of relative motion let us consider the path of P with regard to Q. It will be easy to see that this corresponds exactly to the following question.

A boat, propelled (relatively to the water) with constant velocity u, *starts from a point* A *in the bank of a river which runs with velocity* v *parallel to* Qx, *and tends continually to the point* Q, *on the other bank, directly opposite to* A; *to find its path.*

The constant velocity of the stream in this case communicated to P corresponds to the constant velocity of Q in the last example, but is in the opposite direction. In fact,



if the earth were to be supposed moving in the direction xQ with constant velocity v, the river would be at rest in space,

and the *actual* motions of P and Q would be the same as in the last example. (See § 26.)

To investigate the path, take Q as origin, Qx, QA as the axes. Then the component velocities of P are v parallel to Qx and u along PQ, and the tangent to its path is in the direction of the resultant of these two. Putting θ for PQx,

we have $$\frac{dx}{dt} = v - u\cos\theta, \text{ and } \frac{dy}{dt} = -u\sin\theta,$$

whence $$\frac{dy}{dx} = -\frac{u\sin\theta}{v - u\cos\theta} = -\frac{\sin\theta}{e - \cos\theta}$$

$$= -\frac{y}{e\sqrt{(x^2+y^2)} - x}.$$

This, being a homogeneous equation, is easily integrated and we have, taking $x = 0$, $y = a$, as co-ordinates of A,

$$\frac{y^{1+e}}{a^e} = \sqrt{(x^2+y^2)} - x \;\ldots\ldots\ldots\ldots\ldots(1),$$

or $$2x = a^e y^{1-e} - a^{-e} y^{1+e},$$

or $$\left(\frac{r\sin\theta}{a}\right)^e = \frac{1-\cos\theta}{\sin\theta},$$

in polar co-ordinates. This evidently gives a parabola about Q as focus, if $e = 1$.

[*Note.* The student is not unlikely to be led into a curious error in looking at this problem from a geometrical point of view. Thus, the velocity along PQ is always in a definite ratio to that in MP produced; why is not the path *always* a conic section of which Q is a focus? The idea is completely erroneous (as in fact the above investigation shews), but it forms the very best training in a science like Kinematics to seek to explain such difficulties without any aid from analysis.]

36. *To find the time of crossing the stream.*

This may easily be effected by considering the actual velocity parallel to the axis of y:

$$\frac{dy}{dt} = -u \sin\theta$$

$$= -u \frac{y}{\sqrt{(x^2+y^2)}}.$$

Now taking quotients of y^2 by both sides of (1),

$$a^e y^{1-e} = \sqrt{(x^2+y^2)} + x.$$

Hence $$2\sqrt{(x^2+y^2)} = a^e y^{1-e} + a^{-e} y^{1+e};$$

and therefore $\dfrac{dy}{y}(a^e y^{1-e} + a^{-e} y^{1+e}) = -2udt.$

Taking the integral from a to 0, and putting T_1 for the time of crossing,

$$\frac{a}{1-e^2} = uT_1; \text{ or } T_1 = \frac{au}{u^2-v^2}.$$

But, if there had been no current, we should have had for the time of crossing,

$$T_0 = \frac{a}{u}; \text{ whence } \frac{T_1}{T_0} = \frac{u^2}{u^2-v^2}.$$

In the integration we have, of course, $e < 1$, else the boat could not reach Q.

If $e = 1$, the boat will reach the farther bank, but not at Q. The solution of this case presents no special difficulty.

37. If the motion of a point in a plane be considered with reference to a fixed point in that plane, the rate of increase of the angle made by the line joining the two points, with some fixed line in the plane, is called the *Angular Velocity* of the former point about the latter. Unit of angular velocity corresponds to the description of an arc equal to radius in unit of time.

Suppose the above-mentioned angle to be represented by θ at time t; then at time $t + \delta t$ it has the value $\theta + \delta\theta$, and it may be shewn as before (§ 7), that if ω represent the angular velocity required, then

$$\omega = \frac{d\theta}{dt}.$$

Ex. *A point moves with constant velocity* v *in a straight line; to find at any instant its angular velocity about a fixed point whose distance from the straight line is* a.

Taking as initial line the perpendicular from the fixed point on the line of motion, the polar equation of the path is

$$r = a \sec\theta.$$

Also, if $\theta = 0$, when $t = 0$, we have

$$r \sin\theta = vt.$$

Hence, $a \tan\theta = vt$,

and
$$\omega = \frac{d\theta}{dt} = \frac{va}{a^2 + v^2t^2} = \frac{va}{r^2}.$$

38. *A point describes a circle with constant velocity; it is required to find the actual velocity, and the angular velocity (about the centre) in any orthographic projection.*

Let ApA' be an ellipse and APA' the auxiliary circle. Then the former will be the orthographic projection of the latter if its axes be made in the ratio of the cosine of the angle (α) between the planes of projection. Also if PpM be perpendicular to AA', P and p will be corresponding points in the two. Draw the tangents pT, PT; then

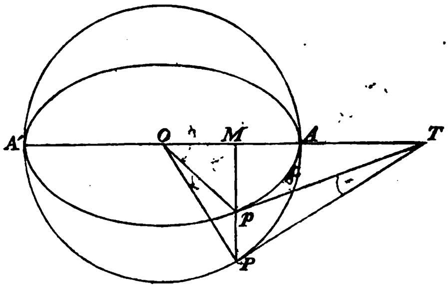
$$\frac{\text{actual velocity at } p}{\text{.....................} \; P} = \frac{pT}{PT}: \text{ and if } TOP = \theta,$$

$$\frac{\text{velocity at } p}{\text{............} \; P} = \frac{\sqrt{(PT^2 \sin^2\theta + PT^2 \cos^2\theta \cos^2\alpha)}}{PT}$$

$$= \sqrt{(\sin^2\theta + \cos^2\theta \cos^2\alpha)}$$

$$= \sqrt{(1 - \sin^2\alpha \cos^2\theta)}.$$

Now, if $TOp = \phi$, $\dfrac{\text{angular velocity at } p}{\text{.................. } P} = \dfrac{d\phi}{d\theta}$



$$= \frac{d}{d\theta} \tan^{-1} (\cos\alpha \tan\theta)$$

$$= \frac{\cos\alpha}{\cos^2\theta + \cos^2\alpha \sin^2\theta}$$

$$= \frac{\cos\alpha}{1 - \sin^2\alpha \sin^2\theta}.$$

This is a maximum if $\theta = \dfrac{\pi}{2}$, when its value is $\sec\alpha$,

.... minimum ... $= 0$ .... $\cos\alpha$.

Hence, if ω_1 and ω_2 be the greatest and least angular velocities in the projection,

$\sqrt{(\omega_1\omega_2)}$ is the angular velocity in the original path.

39. Evidently, the product of the radius-vector into the angular velocity is the velocity perpendicular to the radius-vector. (§ 11.) This is to the whole velocity as the perpendicular on the tangent is to the radius-vector; and therefore the product of the square of the radius-vector by the angular velocity is equal to the product of the whole velocity by the perpendicular on the tangent, *i.e.* to the moment of velocity about the pole, § 24, (1).

40. When the angular velocity is variable, its rate of change per unit of time is called the *Angular Acceleration*, and is measured with reference to the same unit angle.

Thus, in the Ex. § 37, the angular acceleration is

$$\frac{d\omega}{dt} = -\frac{2va}{r^3}\frac{dr}{dt} = -\frac{2v^2a}{r^4}\sqrt{r^2-a^2}.$$

41. *The motion of a point in a plane being given with respect to fixed axes, to investigate expressions for its velocity and acceleration relative to axes in the same plane, which revolve about a common origin with constant angular velocity.*

Let ω be this angular velocity; then, if at time $t=0$ the fixed and revolving axes coincide, at time t they will be inclined to one another at an angle ωt. Hence, if x, y, ξ, η be the co-ordinates of the point at time t, referred to the fixed and to the revolving axes respectively, we have by the ordinary formulæ for transformation of co-ordinates

$$\left.\begin{aligned} \xi &= x\cos\omega t + y\sin\omega t \\ \eta &= y\cos\omega t - x\sin\omega t \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

These give, by differentiation,

$$\frac{d\xi}{dt} = \frac{dx}{dt}\cos\omega t + \frac{dy}{dt}\sin\omega t - \omega\,(x\sin\omega t - y\cos\omega t)$$

$$\left.\begin{aligned} &= \frac{dx}{dt}\cos\omega t + \frac{dy}{dt}\sin\omega t + \omega\eta. \\ \text{Similarly,}\quad \frac{d\eta}{dt} &= \frac{dy}{dt}\cos\omega t - \frac{dx}{dt}\sin\omega t - \omega\xi, \end{aligned}\right\} \quad \ldots\ldots\ldots(2),$$

which determine the velocities relative to the revolving axes.

Again,

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= \frac{d^2x}{dt^2}\cos\omega t + \frac{d^2y}{dt^2}\sin\omega t - 2\omega\left(\frac{dx}{dt}\sin\omega t - \frac{dy}{dt}\cos\omega t\right) - \omega^2\xi \\ \frac{d^2\eta}{dt^2} &= \frac{d^2y}{dt^2}\cos\omega t - \frac{d^2x}{dt^2}\sin\omega t - 2\omega\left(\frac{dy}{dt}\sin\omega t + \frac{dx}{dt}\cos\omega t\right) - \omega^2\eta \end{aligned}\right\} (3),$$

or

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= \frac{d^2x}{dt^2}\cos\omega t + \frac{d^2y}{dt^2}\sin\omega t + 2\omega\frac{d\eta}{dt} + \omega^2\xi \\ \frac{d^2\eta}{dt^2} &= \frac{d^2y}{dt^2}\cos\omega t - \frac{d^2x}{dt^2}\sin\omega t - 2\omega\frac{d\xi}{dt} + \omega^2\eta \end{aligned}\right\} \quad \ldots\ldots\ldots(3'),$$

the relative accelerations.

Now the component accelerations along *fixed* axes, with which at the time t the moving axes coincide, are evidently represented by the first two terms of the right-hand sides of these equations; or, in terms of the co-ordinates with respect to the moving axes, by

$$\frac{d^2\xi}{dt^2} - 2\omega\frac{d\eta}{dt} - \omega^2\xi, \text{ and } \frac{d^2\eta}{dt^2} + 2\omega\frac{d\xi}{dt} - \omega^2\eta \ldots\ldots(4).$$

Ex. If the point be at rest, x and y are constant, and

$$\frac{d\xi}{dt} = \omega\eta, \quad \frac{d\eta}{dt} = -\omega\xi.$$

Also
$$\frac{d^2\xi}{dt^2} = -\omega^2\xi, \quad \frac{d^2\eta}{dt^2} = -\omega^2\eta.$$

These expressions are obvious, as in this case the relative motion of the point with respect to the moving axes is a uniform circular motion about the origin, in the *negative* direction, *i.e.* from the axis of η to that of ξ.

42. *Suppose the new axes not to revolve uniformly.*

In this case the investigation is precisely the same as the above, with the exception that θ, a given function of t, must be substituted for ωt. If ω, now no longer constant, be put for $\frac{d\theta}{dt}$, the student will have no difficulty in verifying the following expressions, which take the place of (2), (3′) and (4) of the preceding section.

$$\left.\begin{aligned}\frac{d\xi}{dt} &= \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta + \omega\eta\\ \frac{d\eta}{dt} &= \frac{dy}{dt}\cos\theta - \frac{dx}{dt}\sin\theta - \omega\xi\end{aligned}\right\}\ldots\ldots\ldots\ldots(2_1).$$

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} &= \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta - \omega^2\xi + 2\omega\frac{d\eta}{dt} + \frac{d\omega}{dt}\eta\\ \frac{d^2\eta}{dt^2} &= \frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta - \omega^2\eta - 2\omega\frac{d\xi}{dt} - \frac{d\omega}{dt}\xi\end{aligned}\right\}\ldots(3'_1).$$

$$\frac{d^2\xi}{dt^2} - \omega^2\xi - \frac{1}{\eta}\frac{d}{dt}(\omega\eta^2), \quad \frac{d^2\eta}{dt^2} - \omega^2\eta + \frac{1}{\xi}\frac{d}{dt}(\omega\xi^2)\ldots\ldots(4_1).$$

These expressions might have been deduced at once from the expressions in § 16, by the consideration of relative accelerations as in § 26. Let $OM = \xi$, $MP = \eta$, be the co-ordinates of the point referred to the moving axes. Then, by § 16, the acceleration of M along OM is

$$\frac{d^2\xi}{dt^2} - \omega^2\xi.$$

Also, as MP revolves with angular velocity ω, the acceleration of P relative to M, in the direction perpendicular to MP, is

$$\frac{1}{\eta}\frac{d}{dt}(\omega\eta^2).$$

This is in the direction of the negative part of the axis of ξ. Hence the resolved part parallel to $O\xi$, of the acceleration of P with respect to O, is

$$\frac{d^2\xi}{dt^2} - \omega^2\xi - \frac{1}{\eta}\frac{d}{dt}(\omega\eta^2).$$

43. The principles already enunciated, and the examples given of their application, will suffice for the solution of problems on this part of the subject.

Other examples of the application of these principles, such as the kinematical part of the investigations of the Hodograph, &c., will be more appropriately introduced in future chapters.

EXAMPLES.

(1) A point moves from rest in a given path, and its velocity at any instant is proportional to the time elapsed since its motion commenced; find the space described in a given time.

(2) If a point begin to move with velocity v, and at equal intervals of time a velocity u be communicated to it in the same direction; find the space described in n such intervals.

(3) A man six feet high walks in a straight line at the rate of four miles an hour away from a street lamp, the height of which is 10 feet; supposing the man to start from the lamp-post, find the rate at which the end of his shadow travels, and also the rate at which the end of his shadow separates from himself.

(4) If the position of a point moving in a plane be determined by the co-ordinates ρ and ϕ, ρ being measured from a fixed circle (radius a) along a tangent which has revolved through an angle ϕ from a fixed tangent; investigate the following expressions for the accelerations along and perpendicular to ρ respectively,

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2 + a\frac{d^2\phi}{dt^2}$$

and $$\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\phi}{dt}\right) + a\left(\frac{d\phi}{dt}\right)^2.$$

(5) Prove that it is not possible for a point to move so that its velocity in any position may be proportional to the length of the path which it has described from rest: also that if its velocity be proportional to the space it has to describe, however small, it will never accomplish it.

(6) The velocity of a point parallel to each of three rectangular axes is proportional to the product of the other two co-ordinates; what are the equations of the path, and what is the time of describing a given portion when the curve passes through the origin?

(7) A point moves in a plane, so that its velocities parallel to the axes of x and y are

$$u + ey \text{ and } v + \epsilon x \text{ respectively,}$$

shew that it moves in a conic section.

(8) Two points are moving with constant velocity in two straight lines, 1st in a plane, 2nd in space; given the initial circumstances, find when they are nearest to each other. Shew also that in both cases the relative path is a straight line, described with constant velocity.

(9) A number of points are moving with constant velocity in straight lines in space; determine the motion of their common centre of inertia. (§ 58.)

(10) A cannon-ball is moving in a direction making an acute angle θ with a line drawn from the ball to an observer; if V be the velocity of sound, and nV that of the ball, prove that the whizzing of the ball at different points of its course will be heard in the order in which it is produced, or in the reverse order, according as $n <> \sec\theta$.

(11) A particle, projected with a velocity u, is acted on by a force, which produces a constant acceleration f, in the plane of motion, inclined at a constant angle α to the direction of motion. Obtain the intrinsic equation of the curve described, and shew that the particle will be moving in the opposite direction to that of projection at the time

$$\frac{u}{f\cos\alpha}\left(e^{\pi\cot\alpha}-1\right).$$

(12) Shew that any infinitely small motion given to a plane figure in its own plane is equivalent to a rotation through an infinitely small angle about some point in the figure.

Hence shew that the relative motion of two figures in a plane may be produced by rolling a curve fixed to one figure on a curve fixed to the other figure. (These curves are called Centroids.)

(13) The highest point of the wheel of a carriage rolling on a road moves twice as fast as each of two points in the rim whose distance from the ground is half the radius of the wheel.

(14) A rod of given length moves with its ends in two given lines which intersect; shew how to draw a tangent to the path described by any point of the rod.

(15) Investigate the position of the instantaneous centre about which the rod is turning, and apply this also to solve the preceding question.

(16) One circle rolls on another whose centre is fixed. From the initial and final positions of a diameter in each

determine what portions of their circumferences have been in contact.

(17) One point describes the diameter AB of a circle with constant velocity, and another the semi-circumference AB from rest with constant tangential acceleration; they start together from A and arrive together at B; shew that the velocities at B are as $\pi : 1$.

(18) In the example of § 33 find in the case of $e < 1$ the length of time occupied in the pursuit.

(19) In the example of § 34 find the greatest distance the boat is carried down the stream, and shew that when it is in that position its velocity is $\sqrt{(u^2 - v^2)}$.

When $u = v$, shew directly that the curve described is a parabola.

(20) Shew that if ρ be the radius of curvature of the curve of pursuit, we have in the figure of § 33,

$$\rho = \frac{PQ^2}{ePM}.$$

(21) In the case of a boat propelled with velocity u relatively to the water in a stream running with velocity v, shew that the boat passes from one given point to another in the least possible time when its actual path is a straight line.

(22) The velocity of a stream varies as the distance from the nearest bank; shew that a man attempting to swim directly across will describe two semiparabolas. (Shew that the sub-normal is constant.) Find by how much the mean velocity is increased.

(23) A point moves with constant velocity in a circle; find an expression for its angular velocity about any point in the plane of the circle.

(24) If the velocity of a point moving in a plane curve vary as the radius of curvature, shew that the direction of motion revolves with constant angular velocity.

(25) Two bevelled wheels roll together; having given the inclinations of the axes of the cones, find their vertical

angles that the wheels may revolve with angular velocities in a given ratio.

(26) Supposing the Earth and Venus to describe in the same plane circles about the Sun as centre; investigate an expression for the angular velocity of the Earth about Venus in any position, the actual velocities being inversely as the square roots of their distances from the Sun.

(27) A particle moving uniformly round the circular base of an oblique cone is projected by generating lines on a subcontrary section; find its angular velocity about the centre of the latter.

(28) If ξ, η denote the co-ordinates of a moving point referred to two axes, one of which is fixed and the other rotates with constant angular velocity ω, prove that its component accelerations parallel to these axes are

$$\frac{d^2\xi}{dt^2} - 2\omega \operatorname{cosec} \omega t \frac{d\eta}{dt},$$

$$\frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega \cot \omega t \frac{d\eta}{dt}.$$

(29) Two lines are moving in their own plane about their point of intersection with constant angular velocities ω, ω'; if the co-ordinates of a moving point referred to them be x, y at a time t, prove that its accelerations parallel to the axes are

$$\frac{d^2x}{dt^2} - \omega^2 x - 2\omega \cot (\omega' - \omega)\, t \frac{dx}{dt} - 2\omega' \operatorname{cosec} (\omega' - \omega)\, t \frac{dy}{dt},$$

$$\frac{d^2y}{dt^2} - \omega^2 y - 2\omega \operatorname{cosec} (\omega' - \omega)\, t \frac{dx}{dt} - 2\omega' \cot (\omega' - \omega)\, t \frac{dy}{dt}.$$

(30) Employ the formulæ of § (30) to trace approximately the form of the path of C about A, when m is nearly, but not exactly, equal to $+n$ or to $-n$.

(31) If an odd number n of rods $OA_1, A_1A_2, A_2A_3, \ldots$ whose lengths are $a, \frac{a}{2}, \frac{a}{3}, \ldots \frac{a}{n}$, are hinged together at $A_1, A_2, \ldots$ and

revolve with constant angular accelerations α, 2α, 3α, ...$n\alpha$, about their extremities $O_1A_1A_2, \ldots A_{n-1}$, shew that the direction of motion of the point A_n at any time is perpendicular to the direction of the middle rod; the motion commencing from rest with the rods in a straight line.

(32) A man is in a boat, on a river, at a distance a from the shore, and b from a fall of water ahead. If the velocity of the stream be V, prove that he cannot escape the fall unless he can row with a velocity $\frac{a}{\sqrt{a^2+b^2}} V$; and that in case he can just row at this pace, the direction in which he must row is at right angles to the line joining his position with the point of the bank opposite the fall. Find also the direction in which he will have the least distance to row to reach the bank, supposing his velocity greater than this minimum.

(33) If a point is moving in a hypocycloid with velocity u; and v, V represent the velocities of the centre of curvature and the centre of the generating circle corresponding to the position of the point, prove that

$$\frac{u^2}{(c-b)^2} + \frac{v^2}{(c+b)^2} = \frac{4V^2}{(c-b)^2},$$

c being the distance between the centres of the generating circles, and b the radius of the moving circle.

(34) N particles are arranged equably along the circumference of a circle of radius a; each continually moves towards the next in order with a constant velocity v; shew that they will all arrive together at the centre of the circle in the time

$$\frac{a}{v} \operatorname{cosec} \frac{\pi}{N}.$$

(35) A point P moves with constant velocity in a circle; Q is a point in the same radius at double the distance from the centre, PR is a tangent at P equal to the arc described by P from the beginning of the motion: shew that the acceleration of the point R is represented in direction and magnitude by RQ.

(36) If a point move in an orbit so that the area described in any time by the radius of curvature is proportional

to that time, prove that the direction of the acceleration of the point is perpendicular to the line joining the point to the corresponding centre of curvature of the evolute, and its magnitude (F) is given by the equation

$$\frac{F^2}{u^2} = c^4 \left\{ \left(\frac{du}{ds}\right)^2 + u^4 \right\},$$

where u is the index of curvature at the point, and c is twice the area described in a unit of time.

(37) A body P is describing an ellipse in any manner: Q is a fixed point on the major-axis and PG the normal at P. Shew that at the moment when G coincides with Q, the angular velocity of P about Q is to its angular velocity about G as CD^2 to CB^2.

(38) A plane is moving about an axis perpendicular to it, and a point is moving in a given curve traced on the plane; in any position ω is the angular velocity of the plane, v the velocity of the particle relative to the plane, r its distance from the axis, p the perpendicular on the tangent, s the arc described along the plane; prove that the acceleration along the tangent to the curve is

$$v\left(\frac{dv}{ds} + p\frac{d\omega}{ds}\right) - \omega^2 r \frac{dr}{ds}.$$

(39) A particle moves on a surface: v, v' are the components of its velocity along the lines of curvature, ρ, ρ' the principal radii of curvature; prove that the acceleration along the normal to the surface $= \dfrac{v^2}{\rho} + \dfrac{v'^2}{\rho'}$.

(40) The intrinsic equation of a curve being $s = f(\phi)$, the curve is described by a point with accelerations X Y parallel to the tangent and normal at the point for which $\phi = 0$; prove that

$$\cos\phi\left(\frac{dY}{d\phi} - 3X\right) - \sin\phi\left(\frac{dX}{d\phi} + 3Y\right) + \frac{f''(\phi)}{f'(\phi)}\,(Y\cos\phi - X\sin\phi) = 0.$$

(41) Obtain expressions for the accelerations of a moving point whose co-ordinates are r, θ, ϕ, (1) in the direction of r, (2) in the direction perpendicular to the radius vector and in the plane of θ, (3) in the direction perpendicular to the plane of θ.

A point describes a rhumb line on a sphere in such a way that its longitude increases uniformly; prove that the resultant acceleration varies as the cosine of the latitude, and that its direction makes with the normal an angle equal to the latitude.

(42) A rigid plane sheet is deprived by guide-pieces of all freedom of motion save parallel to a fixed line in its plane. If it be set in motion by the end of a crank, describing a given path in a given manner and working in a slot of given form cut in the sheet, form the equation of rectilinear motion of the sheet.

(43) Investigate completely the cases of Example (42) when

(*a*) the slot is straight,

(*b*) the slot is a circular arc,

the motion of the crank being circular and uniform.

CHAPTER II.

LAWS OF MOTION.

44. Having, in the preceding chapter, very briefly considered the purely geometrical properties of the motion of a point, we must now treat of the causes which produce various circumstances of motion of a *Particle;* and of the experimental laws, on the assumed truth of which all our succeeding investigations are founded. And it is obvious that we now introduce for the first time the ideas of *Matter*, and of *Force.*

We commence with a few definitions and explanations, necessary to the full enunciation of Newton's Laws and their consequences.

45. The *Quantity of Matter* in a body, or the *Mass* of a body, is proportional to the *Volume* and the *Density* conjointly. The *Density* may therefore be defined as the quantity of matter in unit volume.

If M be the mass, ρ the density, and V the volume, of a homogeneous body, we have at once

$$M = V\rho\,;$$

if we so take our units that unit of mass is the mass of unit volume of a body of unit density.

As will be presently explained, the most convenient unit mass is an *Imperial Pound* of matter.

46. A *Particle* of matter is supposed to be so small that, though retaining its material properties, it may be treated, so

far as its co-ordinates, &c. are concerned, as a geometrical point.

47. The *Quantity of Motion*, or the *Momentum*, of a moving body is proportional to its mass and velocity conjointly.

Hence, if we take as unit of momentum the momentum of a unit of mass moving with unit velocity, the momentum of a mass M moving with velocity v is Mv.

48. *Change of Quantity of Motion*, or *Change of Momentum*, is proportional to the mass moving and the change of its velocity conjointly.

Change of velocity is to be understood in the general sense of § 10. Thus, with the notation of that section, if a velocity represented by OA be changed to another represented by OC, the change of velocity is represented in magnitude and direction by AC.

49. *Rate of Change of Momentum*, or *Acceleration of Momentum*, is proportional to the mass moving and the acceleration of its velocity conjointly. Thus (§ 17) the acceleration of momentum of a particle moving in a curve is $M\frac{d^2s}{dt^2}$ along the tangent, and $M\frac{v^2}{\rho}$ in the radius of absolute curvature.

50. The *Vis Viva*, or *Kinetic Energy*, of a moving body is proportional to the mass and the square of the velocity, conjointly. If we adopt the same units of mass and velocity as before, there is particular advantage in defining kinetic energy as *half* the product of the mass into the square of its velocity.

51. *Rate of Change of Kinetic Energy* (when defined as above) is the product of the velocity into the component of acceleration of momentum in the direction of motion.

For $$\frac{d}{dt}\left(\frac{Mv^2}{2}\right)=Mv\frac{dv}{dt}=v\left(M\frac{d^2s}{dt^2}\right).$$

52. Matter has the innate property of resisting external influences, so that every body, as far as it can, remains at rest, or moves with constant velocity in a straight line.

This, the *Inertia* of matter, is proportional to the quantity of matter in the body. And it follows that some *cause* is requisite to disturb a body's uniformity of motion, or to change its direction from the natural rectilinear path.

53. *Impressed Force*, or *Force* simply, is any cause which tends to alter a body's natural state of rest, or of uniform motion in a straight line.

The three elements specifying a force, or the three elements which must be known, before a clear notion of the force under consideration can be formed, are, its place of application, its direction, and its magnitude.

54. The *Measure of a Force* is the quantity of motion which it produces in unit of time. According to this method of measurement, *the standard or unit force is that force which, acting on the unit of matter during the unit of time, generates the unit of velocity.*

Hence the British absolute unit force is the force which, acting on one pound of matter for one second, generates a velocity of one foot per second.

[According to the common system followed till lately in mathematical treatises on dynamics, the unit of mass is g times the mass of the standard or unit weight; g being the numerical value of the acceleration produced (in some particular locality) by the earth's attraction on falling bodies. This definition, giving a varying and a very unnatural unit of mass, is exceedingly inconvenient. In reality, standards of weight are *masses*, not *forces*. They are employed primarily in commerce for the purpose of measuring out a definite *quantity* of matter; not an amount of matter which shall be attracted by the earth with a given force.]

55. To render this standard intelligible, all that has to be done is to find how many absolute units will produce, in any particular locality, the same effect as gravity. The way

to do this is to measure the effect of gravity in producing acceleration on a body unresisted in any way. The most accurate method is indirect, by means of the pendulum. The result of pendulum experiments made at Leith Fort, by Captain Kater, is, that the velocity acquired by a body falling unresisted for one second is at that place 32·207 feet per second. The variation in gravity for one degree of difference of latitude about the latitude of Leith is only ·0000832 of its own amount. The average value for the whole of Great Britain differs but little from 32·2; that is, the attraction of gravity on a pound of matter in this country is 32·2 times the force which, acting on a pound for a second, would generate a velocity of one foot per second; in other words, 32·2 is the number of absolute units which measures the weight of a pound. Thus, speaking very roughly, the British absolute unit of force is equal to the weight of about half an ounce.

56. Forces (since they involve only direction and magnitude) may be represented, as velocities are, by vectors, that is, by straight lines drawn in their directions, and of lengths proportional to their magnitudes, respectively.

Also the laws of composition and resolution of any number of forces acting at the same point, are, as we shall presently shew, § 67, the same as those which we have already proved to hold for velocities; so that, with the substitution of force for velocity, § 10 is still true.

57. The *Component* of a force in any direction, sometimes called the *Effective Component* in that direction, is therefore found by multiplying the magnitude of the force by the cosine of the angle between the directions of the force and the component. The remaining component in this case is perpendicular to the other.

It is very generally convenient to resolve forces into components parallel to three lines at right angles to each other; each such resolution being effected by multiplying by the cosine of the angle concerned.

The magnitude of the resultant of two, or of three, forces

in directions at right angles to each other, is the square root of the sum of their squares.

58. The *Centre of Inertia or Mass* of any system of material points whatever (whether rigidly connected with one another, or connected in any way, or quite detached), is a point whose distance from any plane is equal to the sum of the products of each mass into its distance from the same plane divided by the sum of the masses.

The distance from the plane of yz, of the centre of inertia of masses m_1, m_2, etc., whose distances from the plane are x_1, x_2, etc., is therefore

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \text{etc.}}{m_1 + m_2 + \text{etc.}} = \frac{\Sigma\,(mx)}{\Sigma m}.$$

And, similarly, for the other co-ordinates.

Hence its distance from the plane

$$\delta = \lambda x + \mu y + \nu z - a = 0,$$

is

$$D = \lambda\bar{x} + \mu\bar{y} + \nu\bar{z} - a,$$

$$= \frac{\Sigma\,\{m\,(\lambda x + \mu y + \nu z - a)\}}{\Sigma m} = \frac{\Sigma\,(m\delta)}{\Sigma m},$$

as stated above. And its velocity perpendicular to that plane is

$$\frac{dD}{dt} = \frac{1}{\Sigma m}\,\Sigma\left\{m\left(\lambda\frac{dx}{dt} + \mu\frac{dy}{dt} + \nu\frac{dz}{dt}\right)\right\} = \frac{\Sigma\left(m\dfrac{d\delta}{dt}\right)}{\Sigma m},$$

from which, by multiplying by Σm, and noting that δ is the distance of x, y, z from $\delta = 0$, we see that the sum of the momenta of the parts of the system in any direction is equal to the momentum in that direction of the whole mass collected at the centre of mass.

59. By introducing, in the definition of moment of velo-

city (§ 21), the mass of the moving body as a factor, we have an important element of dynamical science, the *Moment of Momentum*. The laws of composition and resolution are the same as those already explained.

60. A force is said to *do Work* if it moves the body to which it is applied, and the work done is measured by the resistance overcome, and the space through which it is overcome, conjointly.

Thus, in lifting coals from a pit, the amount of work done is proportional to the weight of the coals lifted; that is, to the force overcome in raising them; and also to the height through which they are raised. The unit for the measurement of work, adopted in practice by British engineers, is that required to overcome the weight of a pound through the height of a foot, and is called a foot-pound.

In purely scientific measurements, the unit of work is not the foot-pound, but the absolute unit force (§ 54) acting through unit of length.

If the weight be raised obliquely, as, for instance, along a smooth inclined plane, the distance through which the force has to be overcome is increased in the ratio of the length to the height of the plane; but the force to be overcome is not the whole weight, but only the resolved part of the weight parallel to the plane; and this is less than the weight in the ratio of the height of the plane to its length. By multiplying these two expressions together, we find, as we might expect, that the amount of work required is unchanged by the substitution of the oblique for the vertical path.

61. Generally, if s be an arc of the path of a particle, S the tangential component of the applied forces, the work done on the particle between any two points of its path is

$$\int S ds,$$

taken between limits corresponding to the initial and final positions.

Referred to rectangular co-ordinates, it is easy to see, by the law of resolution of forces, § 67, that this becomes

$$\int\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)ds.$$

Thus it appears that, for any force, the work done during an indefinitely small displacement of the point of application is the product of the resolved part of the force in the direction of the displacement into the displacement.

From this it follows that, if the motion of a body be always perpendicular to the direction in which a force acts, such a force does no work. Thus the mutual normal pressure between a fixed and a moving body, the tension of the cord to which a pendulum bob is attached, the attraction of the sun on a planet if the planet describe a circle with the sun in the centre, are all cases in which no work is done by the force.

In fact the geometrical condition that the resultant of X, Y, Z shall be perpendicular to ds is

$$X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}=0,$$

and this makes the above expression for the work vanish.

62. Work done on a body by a force is always shewn by a corresponding increase of kinetic energy, if no other forces act on the body which can do work or have work done against them. If work be done against any forces, the increase of kinetic energy is less than in the former case by the amount of work so done. In virtue of this, however, the body possesses an equivalent in the form of *Potential Energy*, if its physical conditions are such that these forces will act equally, and in the same directions, when the motion of the system is reversed. Thus there may be no change of kinetic energy produced, and the work done may be wholly stored up as potential energy.

Thus a weight requires work to raise it to a height, a spring requires work to bend it, air requires work to com-

press it, etc.; but a raised weight, a bent spring, compressed air, etc., are *stores* of energy which can be made use of at pleasure.

These definitions being premised, we give Newton's *Laws of Motion.*

63. LAW I. *Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by forces to change that state.*

We may logically convert the assertion of the first law of motion as to velocity into the following statements :—

The times during which any particular body, not compelled by force to alter the speed of its motion, passes through equal distances, are equal. And, again—Every other body in the universe, not compelled by force to alter the speed of its motion, moves over equal distances in successive intervals, during which the particular chosen body moves over equal distances.

64. The first part merely expresses the convention universally adopted for the measurement of *Time.* The earth, in its rotation about its axis, presents us with a case of motion in which the condition of not being compelled by force to alter its speed, is more nearly fulfilled than in any other which we can easily or accurately observe. Hence the numerical measurement of time practically rests on defining *equal intervals of time,* as *times during which the earth turns through equal angles.* This is, of course, a mere convention, and not a law of nature; and, as we now see it, is a part of Newton's first law.

The remainder of the law is not a convention, but a great truth of nature, which we may illustrate by referring to small and trivial cases as well as to the grandest phenomena we can conceive.

65. LAW II. *Change of motion is proportional to the force, and takes place in the direction of the straight line in which the force acts.*

We have considered change of velocity, or acceleration,

as a purely geometrical quantity, and have seen how it may be at once inferred from the given initial and final velocities of a body. By the definition of motion, or quantity of motion (§ 47), we see that, if we multiply the change of velocity, thus geometrically determined, by the mass of the body, we have the change of motion (§ 48) referred to in Newton's law as the measure of the force which produces it.

It is to be particularly noticed, that in this statement there is nothing said about the actual motion of the body before it was acted on by the force: it is only the *change* of motion that concerns us. Thus the same force will produce precisely the same change of motion in a body, whether the body be at rest, or in motion with any velocity whatever.

66. Again, it is to be noticed that nothing is said as to the body being under the action of *one* force only; so that we may logically put part of the second law in the following (apparently) amplified form:—

When any forces whatever act on a body, then, whether the body be originally at rest or moving with any velocity and in any direction, each force produces in the body the exact change of motion which it would have produced if it had acted singly on the body originally at rest.

67. A remarkable consequence follows immediately from this view of the second law. Since forces are measured by the changes of motion they produce, and their directions assigned by the directions in which these changes are produced; and since the changes of motion of one and the same body are in the directions of, and proportional to, the changes of velocity—a single force, measured by the resultant change of velocity, and in its direction, will be the equivalent of any number of simultaneously acting forces. Hence

The resultant of any number of forces (applied at one point) is to be found by the same geometrical process as the resultant of any number of simultaneous velocities.

From this follows at once (§ 10) the construction of the *Parallelogram of Forces* for finding the resultant of two

forces acting at the same point, and the *Polygon of Forces* for the resultant of any number of forces acting at a point. And, so far as a single particle is concerned, we have at once the whole subject of Statics.

68. The second law gives us the means of measuring force, and also of measuring the mass of a body.

For, if we consider the actions of various forces upon the same body for equal times, we evidently have changes of velocity produced, which are *proportional to* the forces. The changes of velocity, then, give us in this case the means of comparing the magnitudes of different forces. Thus the velocities acquired in one second by the same mass (falling freely) at different parts of the earth's surface, give us the relative amounts of the earth's attraction at these places.

Again, if equal forces be exerted on different bodies, the changes of velocity produced in equal times must be *inversely* as the masses of the various bodies. This is approximately the case, for instance, with trains of various lengths drawn by the same locomotive.

Again, if we find a case in which different bodies, each acted on by a force, acquire in the same time the same changes of velocity, the forces must be proportional to the masses of the bodies. This, when the resistance of the air is removed, is the case of falling bodies; and from it we conclude that *the weight of a body in any given locality, or the force with which the earth attracts it, is proportional to its mass.* The student must be careful to observe that this is no mere truism, but is an important part of the grand *Law of Gravitation.* Gravity is not, like magnetism for instance, a force depending on the *quality* as well as the *quantity* of matter in a particle.

69. It appears, lastly, from this law, that every theorem of Kinematics connected with acceleration has its counterpart in Kinetics. Thus, for instance (§ 18), we see that the force, under which a particle describes any curve, may be resolved into two components, one in the tangent to the curve, the other *towards* the centre of curvature; their magnitudes being the acceleration of momentum, and the product of the momentum into the angular velocity about

the centre of curvature, respectively. In the case of uniform motion, the first of these vanishes, or, the whole force is perpendicular to the direction of motion. When there is no force perpendicular to the direction of motion, there is no curvature, or the path is a straight line.

Hence, if we resolve the forces, acting on a particle of mass m whose co-ordinates are x, y, z, into the three rectangular components X, Y, Z; we have

$$m\frac{d^2x}{dt^2} = X,\ m\frac{d^2y}{dt^2} = Y,\ m\frac{d^2z}{dt^2} = Z.$$

In many of the future chapters these equations will be somewhat simplified by assuming unity as the mass of the moving particle. When this cannot be done, it is sometimes convenient to assume X, Y, Z as the component forces *on unit mass*, and the previous equations become

$$m\frac{d^2x}{dt^2} = mX,\ \&c.;$$

from which m may of course be omitted.

[Some confusion is often introduced by the division of forces into "*accelerating*" and "*moving*" forces; and it is even stated occasionally that the former are of *one*, and the latter of *four* linear dimensions. The fact, however, is that an equation such as

$$\frac{d^2x}{dt^2} = X$$

may be interpreted either as dynamical, or as merely kinematical. If kinematical, the meanings of the terms are obvious; if dynamical, the unit of mass must be understood as a factor on the left-hand side, and in that case X is the x-component, per unit of mass, of the whole force exerted on the moving body.]

If there be no acceleration, we have of course equilibrium among the forces. Hence the equations of motion of a particle are changed into those of equilibrium by putting

$$\frac{d^2x}{dt^2} = 0,\ \&c.$$

70. We have, by means of the first two laws, arrived at a *definition* and a *measure* of force; and have found how to compound, and therefore how to resolve, forces; and also how to investigate the conditions of equilibrium or motion of a single particle subjected to given forces. But more is required before we can completely understand the more complex cases of motion, especially those in which we have mutual actions between or amongst two or more bodies; such as, for instance, tensions or pressures or transference of energy in any form. This is perfectly supplied by

71. LAW III. *To every action there is always an equal and contrary reaction: or, the mutual actions of any two bodies are always equal and oppositely directed in the same straight line.*

If one body presses or draws another, it is pressed or drawn by this other with an equal force in the opposite direction. If any one presses a stone with his finger, his finger is pressed with an equal force in the opposite direction by the stone. A horse, towing a boat on a canal, is dragged backwards by a force equal to that which he impresses on the towing-rope forwards. By whatever amount, and in whatever direction, one body has its motion changed by impact upon another, this other body has its motion changed by the same amount in the opposite direction; for at each instant during the impact they exerted on each other equal and opposite pressures. When neither of the two bodies has any rotation, whether before or after impact, the changes of velocity which they experience are inversely as their masses. When one body attracts another from a distance, this other attracts it with an equal and opposite force.

72. We shall for the present take for granted, that the mutual action between two particles may in every case be imagined as composed of equal and opposite forces in the straight line joining them, two such equal and opposite forces constituting a "stress" between the particles. From this it follows that the sum of the quantities of motion, parallel to any fixed direction, of the particles of any system influencing one another in any possible way, remains unchanged by their mutual action; also that the sum of the moments of momentum of all the particles round any line in a fixed direction in

space, and passing through any point moving uniformly in a straight line in any direction, remains constant. From the first of these propositions we infer that the centre of mass of any system of mutually influencing particles, if in motion, continues moving uniformly in a straight line, unless in so far as the direction or velocity of its motion is changed by stresses between the particles and some other matter not belonging to the system; also that the centre of mass of any system of particles moves just as all their matter, if concentrated in a point, would move under the influence of forces equal and parallel to the forces really acting on its different parts. From the second we infer that the axis of resultant rotation through the centre of mass of any system of particles, or through any point either at rest or moving uniformly in a straight line, remains unchanged in direction, and the sum of moments of momenta round it remains constant if the system experiences no force from without. [This principle is sometimes called *Conservation of Areas*, a very misleading designation.] These results will be deduced analytically in Chap. XII.

73. What precedes is founded upon Newton's own comments on the third law, and the actions and reactions contemplated are the pairs of forces, of which each pair constitutes a "stress." In the scholium appended, he makes the following remarkable statement, introducing another specification of actions and reactions subject to his third law:—

Si æstimetur agentis actio ex ejus vi et velocitate conjunctim; et similiter resistentis reactio æstimetur conjunctim ex ejus partium singularum velocitatibus et viribus resistendi ab earum attritione, cohæsione, pondere, et acceleratione oriundis; erunt actio et reactio, in omni instrumentorum usu, sibi invicem semper æquales.

In a previous discussion Newton has shewn what is to be understood by the velocity of a force or resistance; *i.e.*, that it is the velocity of the point of application of the force *resolved in the direction of the force.* Bearing this in mind, we may read the above statement as follows:—

If the Action of an agent be measured by its amount and its velocity conjointly; and if, similarly, the Reaction of the resist-

ance be measured by the velocities of its several parts and their several amounts conjointly, whether these arise from friction, cohesion, weight, or acceleration;—Action and Reaction, in all combinations of machines, will be equal and opposite.

74. Newton here points out that resistances against acceleration are to be reckoned as reactions equal and opposite to the actions by which the acceleration is produced. Thus, if we consider any one material point of a system, its reaction against acceleration must be equal and opposite to the resultant of the forces which that point experiences, whether by the actions of other parts of the system upon it, or by the influence of matter not belonging to the system. In other words, it must be in equilibrium with these forces. Hence Newton's view amounts to this, that all the forces of the system, with the reactions against acceleration of the material points composing it, form groups of equilibrating systems for these points considered individually. Hence, by the principle of superposition of forces in equilibrium, all the forces acting on points of the system form, with the reactions against acceleration, an equilibrating set of forces on the whole system. This is the celebrated principle first explicitly stated and very usefully applied by D'Alembert in 1742 and still known by his name.

Newton in the sentence just quoted lays, in an admirably distinct and compact manner, the foundations of the abstract theory of *Energy*, which recent experimental discovery has raised to the position of the grandest of known physical laws. He points out, however, only its application to mechanics. The *actio agentis*, as he defines it, which is evidently equivalent to the product of the effective component of the force, into the velocity of the point at which it acts, is simply, in modern English phraseology, the rate at which the agent works, called the Power of the agent. The subject for measurement here is precisely the same as that for which Watt, a hundred years later, introduced the practical unit of a "*Horse-power*," or the rate at which an agent works when overcoming 33,000 times the weight of a pound through the distance of a foot in a minute; that is, producing 550 foot-pounds of work per second. The unit, however, which is most generally convenient is that which Newton's definition implies, namely, the

rate of doing work in which the unit of work or energy is produced in the unit of time.

75. Looking at Newton's words in this light, we see by § 51 that they may be logically converted into the following form:—

"Work done on any system of bodies (in Newton's statement, the parts of any machine) has its equivalent in work done against friction, molecular forces, or gravity, if there be no acceleration; but if there be acceleration, part of the work is expended in overcoming the resistance to acceleration, and the additional kinetic energy developed is equivalent to the work so spent."

When part of the work is done against molecular forces, as in bending a spring; or against gravity, as in raising a weight; the recoil of the spring, and the fall of the weight, are capable, at any future time, of reproducing the work originally expended (§ 62). But in Newton's day, and long afterwards, it was supposed that work was *absolutely lost* by friction.

76. If a system of bodies, given either at rest or in motion, be influenced by no forces from without, the sum of the kinetic energies of all its parts is augmented in any time by an amount equal to the whole work done in that time by the mutual actions, which we may imagine as acting between its points. When the lines in which these actions act remain all unchanged in length, the forces do no work, and the sum of the kinetic energies of the whole system remains constant. If, on the other hand, one of these lines varies in length during the motion, the mutual actions in it will do work, or will consume work, according as the distance varies with or against them.

77. Experiment has shewn that the mutual actions between the parts of any system of natural bodies always perform, or always consume, the same amount of work during any motion whatever, by which the system can pass from one particular configuration to another: so that each configuration corresponds to a definite amount of kinetic energy. [For the apparent violation of this by friction, impact, &c., see § 78*.]

Hence no arrangement is possible, in which a gain of kinetic energy can be obtained when the system is restored to its initial configuration. In other words, "*the Perpetual Motion is impossible.*"

78. The *potential energy* (§ 62) of such a system, in the configuration which it has at any instant, is the amount of work that its mutual forces perform during the passage of the system from any one chosen configuration to the configuration at the time referred to. It is generally convenient so to fix the particular configuration, chosen for the zero of reckoning of potential energy, that the potential energy in every other configuration practically considered shall be positive.

To put this in an analytical form, we have merely to notice that by what has just been said, the value of

$$\Sigma \int \left(X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} \right) ds$$

is independent of the paths pursued from the initial to the final positions, and therefore that

$$\Sigma \, (Xdx + Ydy + Zdz)$$

is a complete differential. If, in accordance with what has just been said, this be called $-dV$, V is the potential energy, and

$$X_1 = -\frac{dV}{dx_1}, \; \ldots\ldots\ldots$$

Also, by the second law of motion, if m be the mass of a particle of the system whose co-ordinates are x, y, z, we have

$$m_1 \frac{d^2 x_1}{dt^2} = X_1, \; \&\text{c.} = \&\text{c.}$$

$$\text{and } \Sigma \left\{ m \left(\frac{dx}{dt} \frac{d^2x}{dt^2} + \frac{dy}{dt} \frac{d^2y}{dt^2} + \frac{dz}{dt} \frac{d^2z}{dt^2} \right) \right\} dt = \Sigma \, (Xdx + Ydy + Zdz)$$
$$= -dV.$$

The integral is

$$\frac{1}{2} \Sigma \, (mv^2) + V = H,$$

that is, *the sum of the kinetic and potential energies is constant.* This is called the *Conservation of Energy.*

In abstract dynamics, with which alone this treatise is concerned, there is loss of energy by friction, impact, &c. This we simply leave as loss, to be afterwards accounted for in physics.

78*. [The theory of energy cannot be completed until we are able to examine the physical influences which accompany loss of energy. We then see that in every case in which energy is lost by resistance, heat is generated; and we learn from Joule's investigations that the quantity of heat so generated is a perfectly definite equivalent for the energy lost. Also that in no natural action is there ever a development of energy which cannot be accounted for by the disappearance of an equal amount elsewhere by means of some known physical agency. Thus we conclude that, if any limited portion of the material universe could be perfectly isolated, so as to be prevented from either giving energy to, or taking energy from, matter external to it, the sum of its potential and kinetic energies would be the same at all times. But it is only when the inscrutably minute motions among small parts, possibly the ultimate molecules of matter, which constitute light, heat, and magnetism; and the intermolecular forces of chemical affinity; are taken into account, along with the palpable motions and measurable forces of which we become cognizant by direct observation, that we can recognise the universally conservative character of all natural dynamic action, and perceive the bearing of the principle of reversibility on the whole class of natural actions involving resistance, which seem to violate it. It is not consistent with the object of the present work to enter into details regarding transformations of energy. But it has been considered advisable to introduce the very brief sketch given above, not only in order that the student may be aware, from the beginning of his reading, what an intimate connection exists between Dynamics and the modern theories of Heat, Light, Electricity, &c.; but also that we may be enabled to use such terms as "*potential energy*," &c. instead of the unnatural "*Force-functions*," &c. which disfigure some of the modern analytical treatises on our subject.]

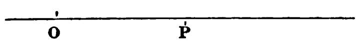
CHAPTER III.

RECTILINEAR MOTION.

79. THE simplest case of motion of a particle which we have to consider is that in a straight line. This may be caused by the applied force acting at every instant in the direction of motion; or the particle may be supposed to be constrained to move in a straight line by being enclosed in a straight tube of indefinitely small bore. As already mentioned, § 69, we shall in every case suppose the mass of the particle to be unity.

80. *A particle moves in a straight line, under the action of any forces, whose resultant is in that line; to determine the motion.*

Let P be the position of the particle at any time t, f the resultant acceleration along OP, O being a fixed point in the line of motion.



Let $OP = x$, then the equation of motion is (by § 69)

$$\frac{d^2x}{dt^2} = f.$$

In this equation f may be given as a function of x, of $\frac{dx}{dt}$, or of t, or of any two or all three combined; but in any case the first and second integrals of the equation (if they can be obtained) will give $\frac{dx}{dt}$ and x in terms of t; that is, the position and velocity of the particle at any instant will be known.

The only one of these cases which we will now consider is that in which f is given as a function of x; those in which f is a function of $\frac{dx}{dt}$, or of $\frac{dx}{dt}$ and x, being reserved for the

Chapter on Motion in a Resisting Medium: while those in which f involves t explicitly possess little interest, as they cannot be procured except by spécial adaptations; and can even then appear only in an incomplete statement of the circumstances of the particular arrangement.

The simplest supposition we can make is that f is constant.

81. *A particle, projected from a given point with a given velocity, is acted on by a constant force in the line of its motion; to determine the position and velocity of the particle at any time.*

Let A be the initial position of the particle, P its position at any time t, v its velocity at that time, and f the constant

acceleration of its velocity. Take any fixed point O in the line of motion as origin, and let $OA = a$, $OP = x$. The equation of motion is

$$\frac{d^2x}{dt^2} = f \quad \text{......................(1).}$$

Integrating once, we have

$$\frac{dx}{dt} = v = ft + C,$$

C being a constant to be determined by the initial circumstances of the motion. Suppose the particle projected from A in the positive direction with velocity V, then when $t = 0$, $v = V$; hence $C = V$, and

$$\frac{dx}{dt} = v = V + ft \quad \text{..............(2).}$$

Integrating again,

$$x = C' + Vt + \frac{1}{2}ft^2.$$

But when $t = 0$, $x = a$; hence $C' = a$, and

$$x = a + V_0 t + \frac{1}{2} f t^2 \text{............} (3).$$

Equations (2) and (3) give the velocity and position of the particle in terms of t; and the velocity may be determined in terms of x by eliminating t between them: but the same result will be obtained more directly by multiplying (1) by $\frac{dx}{dt}$ and integrating. This gives the equation of energy

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{1}{2} v^2 = C'' + fx.$$

But when $x = a$, $v = V$; hence $C'' = \frac{V^2}{2} - fa$, and

$$\frac{1}{2} v^2 = \frac{1}{2} V^2 + f(x - a) \text{..............} (4).$$

82. The most important case of the motion of a particle under the action of a constant acceleration in its line of motion is that of gravity. For the weights of bodies at a given latitude may be considered constant at small distances above the Earth's surface, and therefore if we denote the acceleration due to the Earth's attraction by g, and consider the particle to be projected vertically downwards, equations (2), (3), (4) of § 81 become

$$\left.\begin{aligned} v &= V + gt \\ x &= a + Vt + \tfrac{1}{2} g t^2 \\ \frac{1}{2} v^2 &= \frac{1}{2} V^2 + g(x - a) \end{aligned}\right\} \text{..............} (A),$$

x being measured as before from a fixed point O in the line of motion. As a particular instance suppose the particle to be dropped from rest at O. At that instant A coincides with O, and $a = 0$, $V = 0$.

Hence
$$v = gt \qquad\qquad (1),$$
$$x = \tfrac{1}{2}gt^2 \qquad\qquad (2),$$
$$\frac{1}{2}v^2 = gx \qquad\qquad (3).$$

The last of these equations may also be obtained from
$$g = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$
by a single integration.

83. As another particular instance, suppose the particle to be projected vertically upwards. Here it must be remembered that if we measure x upwards from the point of projection, the acceleration tends to diminish x and the equation of motion is
$$\frac{d^2x}{dt^2} = -g.$$

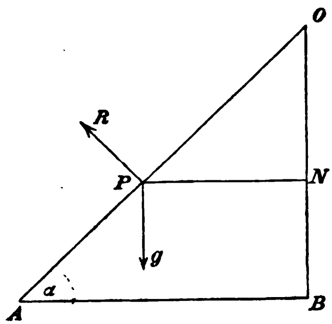
In other respects the solution is the same. Taking, therefore, $a = 0$ in equations (A) and changing the sign of g, we obtain
$$v = V - gt \qquad\qquad (4),$$
$$x = Vt - \frac{1}{2}gt^2 \qquad\qquad (5),$$
$$\frac{1}{2}v^2 = \frac{1}{2}V^2 - gx \qquad\qquad (6).$$

From equation (4) we see that the velocity continually diminishes, and becomes zero when $t = \dfrac{V}{g}$; and from (6) that the height corresponding to $v = 0$, or the greatest height to which the particle will ascend, is $\dfrac{V^2}{2g}$. After this the velocity becomes negative, or the particle begins to descend, and (5) shews that it will return to the point of projection when $t = \dfrac{2V}{g}$, as x then becomes 0; and the velocity with which

it returns to that point is, by (6), equal to the velocity of projection.

84. *A particle descends a smooth inclined plane under the acceleration of gravity, the motion taking place in a vertical plane perpendicular to the intersection of the inclined with any horizontal plane; to determine the motion.*

Let P be the position of the particle at any time t on the inclined plane OA, $OP = x$ its distance from a fixed point O



in the line of motion, and let a be the inclination of OA to the horizontal line AB. The only impressed force on the particle is its weight g which acts vertically downwards, and this may be resolved into two, $g \sin a$ along, and $g \cos a$ perpendicular to OA. Besides these there is the unknown force R, the pressure on the plane, which is perpendicular to OA: but neither this nor the component $g \cos a$ can affect the motion along the plane. The equation of motion is therefore

$$\frac{d^2x}{dt^2} = g \sin a,$$

the solution of which, as $g \sin a$ is constant, is included in that of the proposition of § 82, and all the results for particles moving vertically under the action of gravity will be made to apply to it by writing $g \sin a$ for g. Thus, if the particle

start from rest at O, we get from equations (1), (2), (3) of § 82 by this means,

$$v = g \sin \alpha \, . \, t \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$x = \tfrac{1}{2} g \sin \alpha \, . \, t^2 \ldots\ldots\ldots\ldots\ldots (2),$$

$$\frac{1}{2} v^2 = g \sin \alpha \, . \, x \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

85. Equation (3) proves an important property with regard to the velocity acquired at any point of the descent. For, draw PN parallel to AB, and let it meet the vertical line through O in N, then if v be the velocity at P, we have

$$\frac{1}{2} v^2 = g \sin \alpha \, . \, OP$$

$$= g \, . \, ON.$$

Comparing this with equation (3) of § 82, we see that the velocity at P is the same as that which a particle would acquire by falling freely from rest through the vertical distance ON; in other words the velocity at any point, of a particle sliding down a smooth inclined plane, is that due to the vertical height through which it has descended; a particular case of the conservation of energy.

86. Again from (2) we derive immediately the following curious and useful result.

The times of descent down all chords drawn through the highest or lowest point of a vertical circle are equal.

Let AB be the vertical diameter of the circle, AC any chord through A; join BC; then if T be the time of descent down AC, we have by equation (2) of § 84,

$$AC = \tfrac{1}{2} g T^2 \cos BAC.$$

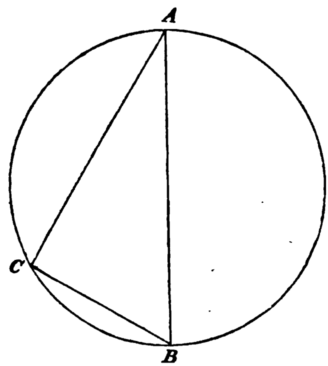
But $AC = AB \cos BAC$; whence

$$AB = \tfrac{1}{2} g T^2,$$

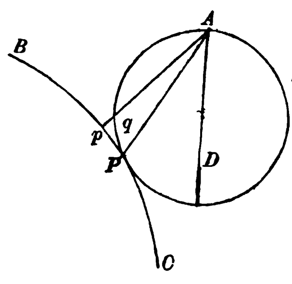
or

$$T = \sqrt{\frac{2AB}{g}},$$

which, being independent of the position of the chord, gives the same time of descent for all.



It may similarly be shewn that the time of descent down all chords through B is the same. In fact parallel chords, drawn through A and B respectively, are of equal length.



To find the straight line of swiftest descent to a given curve from any point in the same vertical plane, all that is required is to draw a circle having the given point as the upper extremity of its vertical diameter, and the smallest which can meet the curve. Hence if BC be the curve, A the point, draw AD vertical; and, with centre in AD, describe a circle passing through A and touching BC. Let P be the point of contact, then AP is the required line. For, if we take any

other point, p, in BC, Ap cuts the circle in some point q, and time down $Ap >$ time down Aq, i.e. $>$ time down AP.

If the given curve be not plane, or if it be required to find the straight line of swiftest descent to a surface, a sphere must be described passing through A, with centre in AD, so as to touch the curve or surface; and the proof is precisely as before.

87. In § 84 we have supposed the inclined plane to be smooth, but the motion will still be constantly accelerated when the plane is rough. For since there is no motion, and therefore no acceleration, perpendicular to OA (see fig. § 84), we must have

$$0 = R - g \cos \alpha. \quad (\S\ 69).$$

If μ then be the coefficient of kinetic friction, which is known by experiment to be independent of the velocity of the particle, the retardation of friction will be μR or $\mu g \cos \alpha$, and the equation of motion will become

$$\frac{d^2x}{dt^2} = g \sin \alpha - \mu g \cos \alpha,$$

the second member still being constant, and the solution therefore similar to those we have already considered.

88. *When a particle moves under an attraction in its line of motion, varying directly as the distance of the particle from a fixed point in that line, to determine the motion.*

Let O be the fixed point, P the position of the particle at any time t, v its velocity at that time, and let $OP = x$. Then

if μ be the acceleration of a particle due to the attraction at a unit of distance from O, which is supposed known, and is called the strength of the attraction, the acceleration at P will be μx, and if it be directed towards O will tend to diminish x. Therefore the equation of motion is

$$\frac{d^2x}{dt^2} = -\mu x,$$

or $$\frac{d^2x}{dt^2} + \mu x = 0 \text{ ....(1)}.$$

Multiplying this equation by $\frac{dx}{dt}$, and integrating, we obtain

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{2}(a^2 - x^2) \text{ ....(2)},$$

the equation of energy. This may be written

$$\frac{dt}{dx} = -\frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{(a^2 - x^2)}};$$

the negative sign being employed if we suppose the motion to be towards O, and a being the distance from the centre at which the velocity is zero. Integrating again,

$$\sqrt{\mu}\,(t - \tau) = \cos^{-1}\frac{x}{a};$$

or $$x = a \cos \sqrt{\mu}\,(t - \tau) \text{ ....(3)},$$

the complete integral of (1); involving two arbitrary constants a and τ, the values of which are to be determined from the initial distance, and the velocity of projection. Thus from (3),

$$\frac{dx}{dt} = v = -\sqrt{\mu}\, a \sin \sqrt{\mu}\,(t - \tau) \text{ ....(4)}.$$

89. Suppose the particle to be projected from A in the positive direction with the velocity V, and let $OA = b$; then when $t = 0$, we have $x = b$, $v = V$; and therefore from (3) and (4)

$$b = a \cos \sqrt{\mu}\tau,$$

$$V = a\sqrt{\mu} \sin \sqrt{\mu}\tau,$$

which determine a and τ, and then (3) and (4) give the position and velocity of the particle at any instant. The velocity in terms of x is obtained directly from (2), for when $x = a$, we have $v = V$; whence $V^2 = \mu(a^2 - b^2)$, and

$$v^2 = V^2 + \mu(a^2 - x^2)$$

90. Equations (3) and (4) give periodical values of x and v, such that all the circumstances of motion are the same at the time $t+\dfrac{2\pi}{\sqrt{\mu}}$ as at the time t. They also shew that the velocity becomes zero when $t=\tau$, and that the corresponding value of x is the greatest possible. Hence the particle will move in the positive direction to a distance a from O, and then begin to return. Also, since when $\sqrt{\mu}(t-\tau)=\pi$, we have $v=0$ again, and $x=-a$, it will pass through O, move to an equal distance on the other side, and so on: the time of a complete oscillation, that is, the time from its leaving any point until it passes through it again in the same direction, being $\dfrac{2\pi}{\sqrt{\mu}}$. This result is remarkable, as it shews that the time of oscillation is independent of the velocity and distance of projection, and depends solely on the strength of the centre of attraction.

The above proposition includes the motion of a particle within a homogeneous sphere of ordinary matter, in a straight bore to the centre. For the attraction of such a sphere on a particle within it is proportional to the distance from the centre, and the equation of motion is therefore the same as that which we have just considered.

91. Suppose O itself to be in motion in the line OA, and let ξ denote its position at time t. The equation of motion is

$$\frac{d^2x}{dt^2}=-\mu(x-\xi),$$

and is integrable when ξ is given in terms of t. This may be at once changed into the equation of relative motion

$$\frac{d^2(x-\xi)}{dt^2}=-\mu(x-\xi)-\frac{d^2\xi}{dt^2},$$

which is the same as when the point O is at rest if $\dfrac{d^2\xi}{dt^2}=0$, i.e. if the velocity of O be constant. If O move with constant acceleration, α, the oscillatory motion will be the same as before, but the mean position will be $\dfrac{\alpha}{\mu}$ behind O.

92. If we have a repulsion from the centre, the equation of motion becomes

$$\frac{d^2x}{dt^2} = \mu x,$$

the integral of which is known to be

$$x = A\epsilon^{t\sqrt{\mu}} + B\epsilon^{-t\sqrt{\mu}};$$

and the motion is not oscillatory. If, when $t = 0$, $x = a$, $v = -a\sqrt{\mu}$, the particle constantly approaches the centre but never reaches it.

93. It is to be remarked that we cannot always apply the same equation of motion to the negative and positive sides of the origin as we have done in the case of § 88. Our being able to do so arises from the fact that the expression, μx, for acceleration changes sign with x; for by looking at the figure it will be seen that when x is negative the acceleration tends to increase x algebraically, and the equation ought properly to be written

$$\frac{d^2x}{dt^2} = +\mu(-x).$$

In general, when the acceleration is proportional to the n^{th} power of the distance, the equations of motion for the positive and negative sides of the origin are respectively

$$\frac{d^2x}{dt^2} = \mu x^n,$$

and
$$\frac{d^2x}{dt^2} = -\mu(-x)^n.$$

The only cases, therefore, in which the same equation of motion will apply to both sides of the origin, occur when n is of the form $\frac{2m+1}{2m'+1}$, where m, m' are any whole numbers including zero, since it is only in these cases that we have

$$-(-x)^n = x^n.$$

94. In other cases the investigation of the motion will generally consist of two parts, one for each side of the origin;

and in one case even when n is of the form $\frac{2m+1}{2m'+1}$ it is necessary to consider these parts separately, because the form of the integral is not sufficiently general to include both. This is when $m=0$ and $m'=-1$, for in that case the equation of motion becomes

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x}.$$

Multiplying this by $\frac{dx}{dt}$ and integrating we have

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2=C-\mu\log x,$$

which becomes impossible when x is negative. But it is evident that we may then write the integral

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2=C-\mu\log(-x),$$

which is, of course, the proper form for the negative side of the origin. These equations cannot generally be integrated farther, but we will shew towards the end of the Chapter how the time of reaching the origin may be determined.

95. *A particle, constrained to move in a straight line, is acted on by an attraction always directed to a point outside the line, and varying directly as the distance of the particle from that point, to determine the motion.*

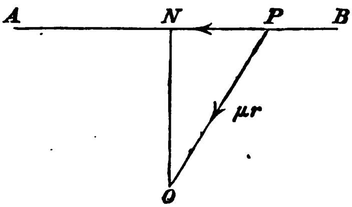
The constraint here contemplated may be conceived by considering the particle either as an indefinitely small ring sliding on a thin smooth wire, or as a material particle sliding in a smooth tube of indefinitely small bore.

Let AB be the straight line, P the position of the particle at any time, O the point to which the attraction on P is always directed. Draw ON perpendicular to AB, and let $NP=x$; then if $OP=r$, and if μ as formerly be the attraction at a unit of distance, the attraction on P along PO is μr. This may be resolved into two, one along and the other perpendicular to AB, of which the latter has no effect on the motion

of the particle. The equation of motion is, therefore, since the acceleration is $\mu r \cos OPN$ or μPN,

$$\frac{d^2x}{dt^2} = -\mu x,$$

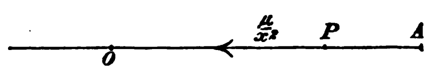
the same as in § 88. The motion of the particle will therefore be oscillatory about N, the time of a complete oscillation



being $\frac{2\pi}{\sqrt{\mu}}$, and all the circumstances of motion the same as for a free particle moving in AB under the action of an equal centre placed at N.

96. *A particle moves in a straight line under the action of an attraction always directed to a point in the line and varying inversely as the square of the distance from that point; to determine the motion.*

Let O be the fixed point, P the position of the particle at



time t, $OP = x$; the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}:$$

μ being, as before, the acceleration at unit distance from O, or the strength of the centre.

Multiplying by $\frac{dx}{dt}$ and integrating, we get

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \mu \int_x^a \frac{dx}{x^2},$$

the equation of energy, supposing the particle to start from rest at a point A distant a from O,

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{v^2}{2} = \mu\left(\frac{1}{x} - \frac{1}{a}\right) \quad \text{.................(1)},$$

which gives the velocity of the particle at any distance x from the origin. Again from (1)

$$\frac{dx}{dt} = -\sqrt{2\mu}\sqrt{\frac{a-x}{ax}},$$

the negative sign being taken, since in the motion towards O, x diminishes as t increases. This gives

$$\frac{dt}{dx} = -\sqrt{\frac{a}{2\mu}} \cdot \frac{x}{\sqrt{(ax-x^2)}}$$

$$= \sqrt{\frac{a}{2\mu}} \cdot \left\{\frac{1}{2}\frac{a-2x}{\sqrt{(ax-x^2)}} - \frac{a}{2}\frac{1}{\sqrt{(ax-x^2)}}\right\}.$$

Integrating, we have

$$t = \sqrt{\frac{a}{2\mu}} \cdot \left\{\sqrt{(ax-x^2)} - \frac{a}{2}\operatorname{vers}^{-1}\frac{2x}{a}\right\}_a^x.$$

Hence $\sqrt{\dfrac{2\mu}{a}}\, t = \sqrt{(ax-x^2)} - \dfrac{a}{2}\operatorname{vers}^{-1}\dfrac{2x}{a} + \dfrac{\pi a}{2}$,

which is the relation between x and t.

97. Putting $x = 0$, we find that the time of arriving at O is

$$\frac{\pi}{2}\sqrt{\frac{a^3}{2\mu}},$$

and (1) shews that the velocity at O is infinite. On this account we are precluded from applying our formulæ to determine the motion after arriving at O; but it is to be observed that, although at any point very near to O there is a very great attraction tending towards O, at the point O itself there is no attraction at all: and therefore the particle, approaching the centre with an indefinitely great velocity, must pass through it. Also, everything being the same at equal distances on either side of the centre, we see that the motion must be checked as rapidly as it was generated, and therefore the particle will proceed to a distance on the other side of S equal to that from which it started. The motion will then continue oscillatory.

98. The above case of motion includes that of a body falling from a great height above the Earth's surface. For a sphere attracts an external particle with an intensity varying inversely as the square of the distance of the particle from its centre, and therefore if x be the distance of a body from the Earth's centre, R the Earth's radius, and g the kinetic measure of gravity on unit of mass at the Earth's surface, the equation of motion will be

$$\frac{d^2x}{dt^2} = -g\frac{R^2}{x^2},$$

the same equation as before, if we write μ for gR^2. The results just obtained will therefore apply to this case. Thus if we wish to find the velocity which a body would acquire in falling to the Earth's surface from a height h above it, we have from (1), putting $\mu = gR^2$,

$$\frac{1}{2}v^2 = gR^2\left(\frac{1}{x} - \frac{1}{R+h}\right);$$

and therefore if V be the velocity when $x = R$, i.e. the required velocity,

$$\frac{1}{2}V^2 = gR\frac{h}{R+h}.$$

If h be small compared with R, this may be written

$$\frac{1}{2}V^2 = gh\left(1 - \frac{h}{R} + \&c.\right),$$

from which we see the amount of error introduced by the ordinary formula, § 82,

$$\frac{1}{2}V^2 = gh.$$

If the fall be from an infinite distance, $a = \infty$, and we have

$$\frac{1}{2}V^2 = gR.$$

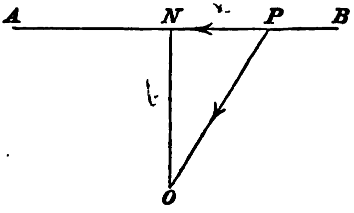
Expressed in terms of the radius and the mean density of the Earth, this becomes

$$\frac{1}{2}V^2 = \frac{4\pi\rho}{3}R^2,$$

which is the kinetic energy acquired by unit of mass falling from rest in infinite space to the Earth's surface.

99. *A particle is constrained to move in a straight line, and is acted on by an attraction, always directed to a point outside that line, and varying inversely as the square of the distance from that point; to determine the motion.*

Let AB be the straight line, P the position of the particle at any time, O the point to which the attraction is always



directed, μ the strength of the centre. Draw ON perpendicular to AB and let $ON = b$, $NP = x$; then the attraction on P along PO is $\frac{\mu}{PO^2}$, and, as in § 95, the only part of this which produces motion is the resolved part along PN. There-

fore the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{OP^2}\cos OPN$$

$$= -\frac{\mu x}{(x^2+b^2)^{\frac{3}{2}}} \dots\dots\dots\dots\dots (1).$$

Multiplying by $\frac{dx}{dt}$ and integrating, we have

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}v^2 = \frac{\mu}{(x^2+b^2)^{\frac{1}{2}}} - \frac{\mu}{(a^2+b^2)^{\frac{1}{2}}},$$

where a is the distance from N to the point where the velocity is zero.

100. This equation cannot generally be integrated farther, but in this and every similar case the integration can be performed if we suppose x always very small. Suppose the particle to have been at rest at N, and to have been slightly displaced from this position of equilibrium, the displacement being so small that throughout the motion $\frac{x^2}{b^2}$ may be neglected in comparison with $\frac{x}{b}$. We have from (1),

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{b^3}\left(1+\frac{x^2}{b^2}\right)^{-\frac{3}{2}}$$

$$= -\frac{\mu x}{b^3}\left(1-\frac{3}{2}\frac{x^2}{b^2} + \&c.\right)$$

$$= -\frac{\mu x}{b^3} \qquad \text{nearly};$$

or

$$\frac{d^2x}{dt^2} + \frac{\mu x}{b^3} = 0,$$

the same form of equation of motion as that of § 88. The motion will therefore be oscillatory, the time of each small oscillation being $2\pi\sqrt{\frac{b^3}{\mu}}$.

101. *A particle moves in a straight line under the action of attraction varying inversely as the* n^th^ *power of the distance of the particle from a fixed point in that line; to determine the motion.*

Measuring x as before, the equation of motion will be

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^n}.$$

Multiplying by $\frac{dx}{dt}$ and integrating, we have

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}v^2 = \frac{\mu}{n-1}\left(\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right) \text{..............(1)},$$

supposing the particle to start from rest at a distance a from the fixed point.

102. This equation cannot generally be integrated farther, but if we suppose the particle to have started from a point at an infinite distance, we have $a = \infty$, and

$$v^2 = \frac{2\mu}{n-1}\,\frac{1}{x^{n-1}},$$

where v is the velocity from infinity, at the distance x.

We have therefore in this particular case

$$\frac{dx}{dt} = \left(\frac{2\mu}{n-1}\right)^{\frac{1}{2}} \frac{1}{x^{\frac{n-1}{2}}}$$

or

$$\frac{dt}{dx} = \left(\frac{n-1}{2\mu}\right)^{\frac{1}{2}} x^{\frac{n-1}{2}}.$$

Integrating this between the limits $x = \alpha$, $x = \beta$, we have for the time of moving from $x = \alpha$ to $x = \beta$,

$$T = \frac{2}{n+1}\left(\frac{n-1}{2\mu}\right)^{\frac{1}{2}} \left(\alpha^{\frac{n+1}{2}} - \beta^{\frac{n+1}{2}}\right).$$

103. To find τ, the time of oscillation, when the amplitude of oscillation is $2a$,

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{n-1}\left(\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right),$$

$$\frac{dt}{dx} = -\sqrt{\frac{n-1}{2\mu}}\sqrt{\frac{a^{n-1}\,x^{n-1}}{a^{n-1} - x^{n-1}}},$$

$$\tau = 4\sqrt{\frac{n-1}{2\mu}}\int_0^a \sqrt{\frac{a^{n-1}\,x^{n-1}}{a^{n-1} - x^{n-1}}}\,dx.$$

Put $$\left(\frac{x}{a}\right)^{n-1} = z; \quad \frac{dx}{dz} = \frac{a}{n-1}\cdot z^{\frac{1}{n-1}-1};$$

and $$\tau = \frac{4a^{\frac{n+1}{2}}}{\sqrt{2\mu(n-1)}}\int_0^1 z^{\frac{1}{n-1}-\frac{1}{2}}(1-z)^{-\frac{1}{2}}\,dz.$$

$$= \frac{4a^{\frac{n+1}{2}}}{\sqrt{2\mu(n-1)}}\,B\left(\frac{1}{n-1}+\frac{1}{2},\ \frac{1}{2}\right)$$

$$= \frac{4a^{\frac{n+1}{2}}}{\sqrt{2\mu(n-1)}}\,\frac{\Gamma\left(\frac{1}{n-1}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n-1}+1\right)}$$

$$= 4a^{\frac{n+1}{2}}\sqrt{n-1}\sqrt{\frac{\pi}{2\mu}}\,\frac{\Gamma\left(\frac{1}{n-1}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n-1}\right)}.$$

104. The above solution fails when $n=1$, but the time of falling to the centre may be found as follows. The equation for this case, as given in § 94, is

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \mu\log\frac{a}{x},$$

since when $x=a$, $\dfrac{dx}{dt}=0$. Hence

$$\sqrt{2\mu}\,\frac{dt}{dx} = -\frac{1}{\sqrt{\log\frac{a}{x}}},$$

the negative sign being taken since x diminishes as t increases. Put T for the required time, then

$$\sqrt{2\mu}\, T = -\int_a^0 \frac{dx}{\sqrt{\log\frac{a}{x}}}.$$

To transform the integral, put $\sqrt{\log\frac{a}{x}} = y$. Then we have

$$x = a\epsilon^{-y^2}, \text{ and } \frac{dx}{dy} = -2a\epsilon^{-y^2} y,$$

and the limits of y are 0 and ∞. Hence

$$\sqrt{2\mu}\,.\,T = 2a\int_0^\infty \epsilon^{-y^2}\,dy$$

$$= 2a\,.\,\tfrac{1}{2}\sqrt{\pi}.$$

Hence $$T = a\sqrt{\frac{\pi}{2\mu}},$$

and is therefore directly as the distance traversed.

105. *A particle is constrained to move in a straight line, and is acted on by an attraction directed to a point not in that line, and expressed by a function $\phi(r)$ of the distance; to determine the time of a small oscillation.*

Employing the same notation as in § 99, the acceleration along PO being $\phi(r)$, its component along PN is $\phi(r)\frac{x}{r}$, therefore the equation of motion is

$$\frac{d^2x}{dt^2} = -\phi(r)\frac{x}{r}.$$

But $$r = \sqrt{(b^2 + x^2)} = b\sqrt{\left(1 + \frac{x^2}{b^2}\right)}$$

$$= b \qquad \text{approximately.}$$

Hence $$\frac{d^2x}{dt^2} + \frac{\phi(b)}{b}x = 0,$$

and therefore by § 90, the time of a small oscillation is

$$2\pi\sqrt{\frac{b}{\phi(b)}}.$$

EXAMPLES.

(1) A body is projected vertically upwards with a velocity which will carry it to a height $2g$; shew that after three seconds it will be descending with a velocity g.

(2) Find the position of a point on the circumference of a vertical circle, in order that the time of rectilinear descent from it to the centre may be the same as the time of descent to the lowest point.

(3) The straight line down which a particle will slide in the shortest time from a given point to a given circle in the same vertical plane, is the line joining the point to the upper or lower extremity of the vertical diameter, according as the point is within or without the circle.

(4) Find the locus of all points from which the time of rectilinear descent to each of two given points is the same. Shew also that in the particular case in which the given points are in the same vertical, the locus is formed by the revolution of a rectangular hyperbola.

(5) Find the line of quickest descent from the focus to a parabola whose axis is vertical and vertex upwards, and shew that its length is equal to that of the latus rectum.

(6) Find the straight line of quickest descent from the focus of a parabola to the curve when the axis is horizontal.

(7) The locus of all points in the same vertical plane for which the least time of sliding down an inclined plane to a circle is constant is another circle.

(8) Two bodies fall in the same time from two given points in space in the same vertical down two straight lines drawn to any point of a surface; shew that the surface is an equilateral hyperboloid of revolution, having the given points as vertices.

(9) Find the form of a curve in a vertical plane, such that if heavy particles be simultaneously let fall from each point of it so as to slide freely along the normal at that point, they may all reach a given horizontal straight line at the same instant.

(10) A semicycloid is placed with its axis vertical and vertex downwards, and from different points in it a number of particles are let fall at the same instant, each moving down the tangent at the point from which it sets out; prove that they will reach the involute (which passes through the vertex) all at the same instant.

(11) A particle moves in a straight line under the action of an attraction varying inversely as the $\left(\frac{3}{2}\right)^{\text{th}}$ power of the distance; shew that the velocity acquired by falling from an infinite distance to a distance a from the centre is equal to the velocity which would be acquired in moving from rest at a distance a to a distance $\frac{a}{4}$.

(12) A particle moves in a straight line from a distance a towards a centre of attraction varying inversely as the cube of the distance; shew that the whole time of descent

$$=\frac{a^2}{\sqrt{\mu}}.$$

(13) A particle is placed at a given point between two centres of equal intensity attracting directly as the distance; to determine the motion and the time of an oscillation.

Let $2a$ be the distance between the centres, x the distance of the particle at any time from the middle point between them, then the equation of motion is

$$\frac{d^2x}{dt^2} = -\mu(a+x) + \mu(a-x)$$
$$= -2\mu x.$$

Hence, the time of an oscillation $= \dfrac{\pi}{\sqrt{(2\mu)}}$.

(14) If a particle begin to move directly towards a fixed centre which repels with an intensity $= \mu$ (distance), and with an initial velocity $= \mu^{\frac{1}{2}}$ (initial distance), prove that it will continually approach the fixed centre, but never attain to it.

(15) A particle acted upon by two centres of attraction, each attracting with an intensity varying inversely as the square of the distance, is projected from a given point between them, to find the velocity of projection that the particle may just arrive at the neutral point of attraction and remain at rest there.

If μ, μ' be the strength of the centres; a_1, a_2 the distances of the point of projection from them; and V the initial velocity; we have

$$V^2 = \frac{(\sqrt{\mu}a_2 - \sqrt{\mu'}a_1)^2}{a_1a_2(a_1+a_2)}.$$

(16) Supposing the earth a homogeneous spheroid of equilibrium, the time of descent of a body let fall from any point P on the surface down a hole bored to the centre C, varies as CP, and the velocity at the centre is constant.

(17) A material particle placed at a centre of attraction varying as the distance, is urged from rest by a constant force which acts for one-sixth of the time of a complete oscillation about the centre, ceases for the same period, and then acts as before, shew that the particle will then be retained at rest, and that the distances moved through in the two periods are equal.

(18) A body moves from rest at a distance a towards a centre of attraction varying inversely as the distance, shew that the time of describing the space between βa and $\beta^n a$ will be a maximum if $\beta = \dfrac{1}{n^{\frac{1}{2(n-1)}}}$.

(19) If the time of a body's descent in a straight line towards a given centre of attraction vary inversely as the square of the distance fallen through, determine the law of the attraction.

(20) Assuming the velocity of a body falling to a centre of attraction to be as $\sqrt{\dfrac{a-x}{x}}$, where a is the initial and x the variable distance from the centre, find the law of the attraction.

(21) Find the time of falling to the centre when the attraction $\propto (\text{dist.})^{-\frac{5}{3}}$.

(22) Shew that the time of descent, to a centre of attraction $\propto (\text{dist.})^{-2}$, through the first half of the initial distance, is to that through the last half as $\pi + 2 : \pi - 2$.

(23) A particle descends to a centre of attraction, intensity $\propto (\text{dist.})^{n}$. Find n so that the velocity acquired from infinity to distance a, shall be equal to that acquired from distance a to distance $\frac{1}{4}a$, from the centre.

(24) A particle is placed at the extremity of the axis of a thin attracting cylinder of infinite length and of radius a, shew that its velocity after describing a space x is proportional to

$$\sqrt{\log \frac{x + \sqrt{(x^2 + a^2)}}{a}}.$$

(25) A particle falls to an infinite homogeneous solid bounded by parallel plane faces, find the time of descent.

(26) Every point of a fine uniform ring repels with an intensity $\propto (\text{dist.})^{-2}$, find the time of a small oscillation in its plane, about the centre.

(27) Shew that a body cannot move so that the velocity shall vary as the distance from the beginning of the motion. And if the velocity vary as the cube root of that distance, determine the acceleration, and the time of describing a given distance.

(28) Shew that the time of quickest descent down a focal chord of a parabola whose axis is vertical is

$$\sqrt{\frac{3^{\frac{3}{2}} l}{g}},$$

where l is the latus rectum.

(29) An ellipse is suspended with its major axis vertical, find the diameter down which a particle will fall in the least time, and the limiting value of the excentricity that this may not be the axis major itself.

(30) Particles slide down chords from a point O to a curved surface, under the attraction of a plane whose attraction is as the distance, and they reach the surface in the same time; shew that the surface is generated by the revolution (about a line whose length is a through O perpendicular to the plane) of the curve whose polar equation about O is

$$\rho \cos\theta = a \{1 - \cos(k \cos\theta)\}.$$

(31) If the particles commence their motion at the surface, and reach O after a given time, the equation of the generating curve is

$$\rho \cos\theta = a \{\sec(k \cos\theta) - 1\}.$$

(32) Prove that the times of falling through a given distance AC towards a centre S, under the action of two attractions, one of which varies as the distance, and the other is constant and equal to the original value of the first, are as the arc (whose versed sine is AC) to the chord, in a circle whose radius is AS.

(33) The earth being supposed a thin uniform spherical shell, in the surface of which a circular aperture of given radius is made, if a particle be dropped from the centre of the aperture, determine its velocity at any point of the descent.

(34) If a particle fall down a radius of a circle under the action of an attraction $\propto (D)^3$ in the centre, and ascend the opposite radius under the action of a repulsion of equal intensity at equal distances from the centre, shew that it will acquire a velocity which is a geometric mean between the radius and the intensity at the circumference.

(35) If a particle fall to a centre of attraction of intensity $\propto (D)$; determine the constant attraction which would produce the effect in the same time, and compare the final velocities.

(36) Find the equation of the curve down each of whose tangents a particle will slide to the horizontal axis in a given time.

(37) A sphere is composed of an infinite number of free particles, equally distributed, which gravitate to each other without interfering; supposing the particles to have no initial velocity, prove that the mean density about a given particle will vary inversely as the cube of its distance from the centre.

(38) Prove that if PQ be a chord of quickest descent from one curve in a vertical plane to another, the tangents at P and Q are parallel and PQ bisects the angles between the normals and the vertical.

(39) A rough horizontal plane has the coefficients of friction at any point proportional to the distance from a fixed point S to which an attraction tends whose intensity is $\mu\,(\text{dist.})^{-3}$, prove that if a particle be placed at a distance $a \tan \alpha$ from S it will arrive at S in time

$$\frac{a^2}{\sqrt{\mu}} \log (\sec 2\alpha),$$

a being the distance at which the particle must be placed so as to be on the point of moving.

(40) If a particle P move from rest under the action of an attraction tending to a point S measured by the acceleration $n^2 SP$, determine the time from rest to rest; and shew that, if a small constant retardation f act through a portion of the path extending equally on each side of S the time will be unaltered, and the diminution of the amplitude of one oscillation will be $\frac{2f}{n^2} \cos n\tau$, τ being the time when the disturbance begins.

(41) A fine thread having two masses each equal to P suspended at its extremities is hung over two smooth pegs in the same horizontal line; a mass Q is then attached to the middle point of the portion of the string between the pegs, and allowed to descend under gravity; shew that the velocity of Q at any depth x below the horizontal line is

$$\sqrt{x^2+a^2}\sqrt{\frac{2g\,(Qx+2Pa-2P\sqrt{x^2+a^2})}{Q\,(x^2+a^2)+2Px^2}}.$$

(42) An elastic string has its ends fastened to the ends of a rod of equal length. The middle point of the string is fastened, and at that point is placed a centre of repulsion, which repels every particle of the rod with an intensity $\frac{\mu}{(\text{dist.})^2}$. The rod is then moved parallel to itself through a distance equal to half its length. If in this position the elasticity of the string is such that the rod is in equilibrium, shew that if slightly displaced perpendicular to its length, the time of a small oscillation

$$=4\pi\sqrt{\frac{a^3}{\mu\,(5+\sqrt{2})}}.$$

(43) A particle moves in a straight line under an attraction to a centre in the straight line $\mu x+2\mu'\frac{x^3}{a^2}$, and starts from rest at a distance a from the centre; shew that after a time t the distance from the centre will be

$$a\,\mathrm{cn}\left(\sqrt{\mu'}\,\frac{at}{a\kappa},\ \kappa\right),$$

where $$\kappa^2=\frac{\mu'a^2}{\mu a^2+2\mu'a^2}.$$

CHAPTER IV.

PARABOLIC MOTION.

106. In this chapter we intend to treat principally of the motion of a free particle which is subject to the action of forces whose resultant is parallel to a given fixed line.

The simplest case of course will be when that resultant is constant. The problem then becomes the determination of the motion of a projectile in vacuo and unresisted, since the attraction of the earth may be considered within moderate limits as constant and parallel to a fixed line. This we will now consider.

107. *A free particle moves under the action of a vertical attraction whose intensity is constant; to determine the form of the path, and the circumstances of its description.*

Taking the axis of x horizontal and in the vertical plane and sense of projection, and that of y vertically upwards, it is evident that the particle will continue to move in the plane of xy, as it is projected in it, and is subject to no force which would tend to withdraw it from that plane.

The equations of motion then are

$$\frac{d^2x}{dt^2}=0,\ \frac{d^2y}{dt^2}=-g,$$

if g be the kinetic measure of the attraction per unit of mass.

Suppose that the point from which the particle is projected is taken as origin, that the velocity of projection is V, and that the direction of projection makes an angle α with the axis of x.

The first and second integrals of the above equations will then be

$$\frac{dx}{dt}=V\cos\alpha,\ \frac{dy}{dt}=V\sin\alpha-gt\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$x=V\cos\alpha\,.\,t,\ y=V\sin\alpha\,.\,t-\tfrac{1}{2}gt^2\ldots\ldots\ldots\ldots\ldots(2).$$

These equations give the co-ordinates of the particle and its velocity parallel to either axis for any assumed value of the time.

Eliminating t between equations (2) we obtain the equation of the trajectory, viz.

$$y = x \tan \alpha - \frac{g}{2V^2 \cos^2 \alpha} x^2 \quad \text{.................(3)},$$

which shews that the particle will move in a parabola whose axis is vertical, and vertex upwards.

108. Equation (3) may be written

$$x^2 - \frac{2V^2 \sin \alpha \cos \alpha}{g} x = -\frac{2V^2 \cos^2 \alpha}{g} y,$$

or $$\left(x - \frac{V^2 \sin \alpha \cos \alpha}{g}\right)^2 = -\frac{2V^2 \cos^2 \alpha}{g}\left(y - \frac{V^2 \sin^2 \alpha}{2g}\right).$$

By comparing this with the equation of a parabola referred to its vertex as origin, we find for the co-ordinates x_0, y_0 of the vertex

$$x_0 = \frac{V^2 \sin \alpha \cos \alpha}{g}, \quad y_0 = \frac{V^2 \sin^2 \alpha}{2g}.$$

Hence we obtain the equation of the directrix

$$y = y_0 + \tfrac{1}{4}\,(\text{parameter}) = \frac{V^2 \sin^2 \alpha}{2g} + \frac{V^2 \cos^2 \alpha}{2g} = \frac{V^2}{2g}.$$

Now if v be the velocity of the particle at any point of its path,

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2, \qquad \text{or by (1)}$$

$$= (V^2 \cos^2 \alpha) + (V^2 \sin^2 \alpha - 2Vg \sin \alpha \,.\, t + g^2 t^2)$$

$$= V^2 - 2g\,(V \sin \alpha \,.\, t - \tfrac{1}{2} g t^2)$$

$$= V^2 - 2gy, \qquad \text{by (2)}.$$

To acquire this velocity in falling from rest, the particle must have fallen (§ 82) through a height $\frac{v^2}{2g}$, or $\frac{V^2}{2g} - y$, i.e. through the distance from the directrix.

109. *To find the time of flight along a horizontal plane.*

Put $y = 0$ in equation (3). The corresponding values of x are 0 and $\frac{2V^2}{g}\sin\alpha\cos\alpha$. But the horizontal velocity is $V\cos\alpha$. Hence the time of flight is $\frac{2V\sin\alpha}{g}$; and, ceteris paribus, varies as the sine of the *elevation* (inclination to the horizon) of the direction of projection.

110. *To find the time of flight along an inclined plane passing through the point of projection.*

Let its intersection with the plane of projection make an angle β with the horizon; it is evident that we have only to eliminate y between (3) and $y = x\tan\beta$.

This gives for the abscissa of the point where the projectile meets the plane,

$$x_1 = \frac{2V^2}{g}(\sin\alpha\cos\alpha - \tan\beta\cos^2\alpha)$$

$$= \frac{2V^2\cos\alpha\sin(\alpha-\beta)}{g\cos\beta}.$$

Hence time of flight

$$= \frac{x_1}{V\cos\alpha} = \frac{2V}{g}\frac{\sin(\alpha-\beta)}{\cos\beta}.$$

111. *To find the direction of projection which gives the greatest range on a given plane.*

The range on the horizontal plane is $\frac{V^2}{g}\sin 2\alpha$. For a given value of V this will be greatest when

$$2\alpha = \frac{\pi}{2}, \text{ or } \alpha = \frac{\pi}{4}.$$

That on the inclined plane is $\frac{x_1}{\cos\beta}$, or

$$\frac{2V^2}{g\cos^2\beta}\cos\alpha\sin(\alpha-\beta).$$

That this may be a maximum for a given value of V we must equate to zero its differential coefficient with respect to α, which gives the equation

$$\cos\alpha\cos(\alpha-\beta)-\sin\alpha\sin(\alpha-\beta)=0,$$

or $$\cos(2\alpha-\beta)=0;$$

whence $$\alpha=\frac{1}{2}\left(\frac{\pi}{2}+\beta\right).$$

Hence the direction of projection required for the greatest range makes with the vertical an angle

$$\frac{\pi}{2}-\alpha=\frac{1}{2}\left(\frac{\pi}{2}-\beta\right),$$

that is, it bisects the angle between the vertical and the plane on which the range is measured.

112. *To find the elevation necessary to the particle's passing through a given point.*

Suppose the point in the axis of x and distant a from the origin. Then we must have

$$\frac{V^2}{g}\sin 2\alpha=a,$$

so that a must not be greater than $\frac{V^2}{g}$.

Let α' be the smallest positive angle whose sine is $\frac{ga}{V^2}$.

The admissible values of α are $\frac{\alpha'}{2}$ and $\frac{\pi-\alpha'}{2}$; so that we see there are two directions in which a particle may be projected so as to reach the given point, and that these are equally inclined to the direction of projection $\left(\alpha=\frac{\pi}{4}\right)$ which gives the greatest range.

Suppose the given point to lie in the plane which makes an angle β with the horizon. Then if its abscissa be a, we must have

$$\frac{2V^2}{g\cos\beta}\cos\alpha\sin(\alpha-\beta)=a.$$

If α', α'' be the two values of α which satisfy this equation, we must have

$$\cos\alpha'\sin(\alpha'-\beta)=\cos\alpha''\sin(\alpha''-\beta);$$

and therefore $$\alpha''-\beta=\frac{\pi}{2}-\alpha',$$

or $$\alpha''-\frac{1}{2}\left(\frac{\pi}{2}+\beta\right)=\frac{1}{2}\left(\frac{\pi}{2}+\beta\right)-\alpha'.$$

Hence, as before, the two directions of projection, which enable the particle to strike a point in a given plane through the point of projection, are equally inclined to the direction of projection required for the greatest range along that plane.

113. *To find the envelop of all the trajectories corresponding to different values of α.*

Differentiating equation (3) with respect to α, we get

$$\sec^2\alpha-\frac{gx}{V^2}\frac{\sin\alpha}{\cos^3\alpha}=0,$$

or $$\tan\alpha=\frac{V^2}{gx}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The elimination of α between (3) and (4) gives us as the equation of the required envelop

$$y=\frac{V^2}{2g}-\frac{gx^2}{2V^2},$$

or $$x^2=-\frac{2V^2}{g}\left(y-\frac{V^2}{2g}\right).$$

This represents a parabola, whose axis is vertical, whose focus is the point of projection, and whose vertex is in the common directrix of the trajectories.

It will easily be seen from what has gone before that there are two directions of projection, so that the particle may pass through any given point within this parabola, only one for a point on it; and of course there is no possibility of its reaching (with the given velocity V) any point without this parabola.

114. By a somewhat simpler method of considering the problem we might easily have arrived at some of the more obvious properties of the trajectory, thus

Take the direction of projection as the axis of x, and the vertical downwards from the point of projection as that of y. By the second law of motion we may consider the velocity due to projection to be maintained constant ($= V$) parallel to the axis of x, while we have in addition parallel to the axis of y the portion due to gravity as investigated in § 82.

Hence $$\left.\begin{aligned} x &= Vt \\ y &= \tfrac{1}{2}gt^2 \end{aligned}\right\} \text{at any time,}$$

and therefore $$x^2 = \frac{2V^2}{g}\,y,$$

the equation of a parabola referred to a diameter and the tangent at its vertex. The distance of the origin from the directrix, being $\frac{1}{4}$th of the coefficient of y, is $\dfrac{V^2}{2g}$, and the velocity due to a fall through that height is as before

$$\sqrt{\left(2g \cdot \frac{V^2}{2g}\right)} = V.$$

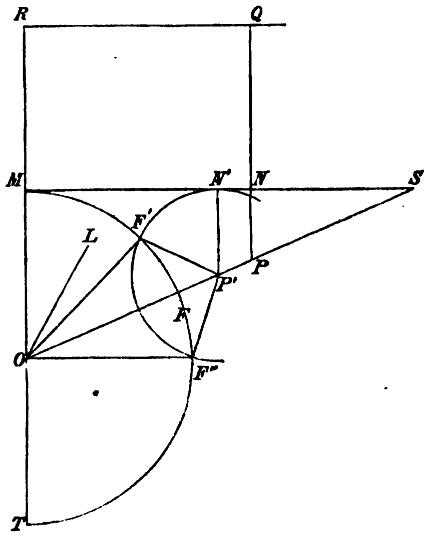
115. Many properties of parabolic motion are more easily obtained by geometry than by analysis. We give a few examples.

Thus suppose O in the figure to be the point of projection, MN the directrix common to the trajectories of all particles projected from O in the plane of the figure with a given velocity, and suppose it be required to determine the direction of projection for the greatest range along the plane OS. Since O is a point in each trajectory and MN the common directrix,

the foci of all possible trajectories lie in the circle $MF'FF''$ described with centre O and touching MN in M.

Take any point in this circle as F', then the path whose focus is F' will intersect OS again in a point P' such that if $P'N'$ be drawn perpendicular to MN, $F'P' = P'N'$. Now in order that P' may be as far as possible from O, at P suppose, it is evident (ex absurdo) that the focus must be taken at the point F where OS meets the circle. But the tangent at O bisects the angle between the diameter MO and the focal distance OF. Hence the direction of projection for the greatest range on an inclined plane bisects the angle between the plane and the vertical.

Again, if with centre P' and radius $P'F'$ an arc be described cutting $F'FF''$ in F'', it is evident that the trajectories



whose foci are F', F'', will intersect OS in the same point P'. Hence, since the directions of projection for these cases will bisect the angles MOF', MOF'' respectively, we see that to strike a given object there are in general two directions of

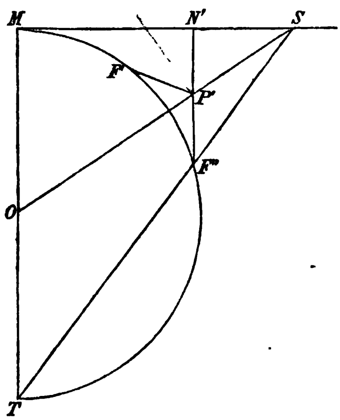
projection, and that these are equally inclined to the direction which gives the greatest range on the plane passing through the object and the point of projection.

Again, for the envelop of all the trajectories. It is evident that P must be a point in the envelop; since it is the ultimate position of P', when the two parabolas which intersect in that point have become indefinitely nearly coincident. Draw PN perpendicular to MN, and produce it till $NQ=FO$. Draw QR parallel to NM, and cutting OM in R. RQ is a fixed line since $RM=MO$, and as $OP=PQ$ we see that the envelop is a parabola whose focus is O and directrix RQ.

It may be seen at once that it touches in P the only trajectory which can pass through that point. For the tangent to either curve at P bisects the angle OPQ or FPN.

116. Ex. *It is required to throw a shell with given velocity so as to strike at right angles an inclined plane through the point of projection.*

The letters being the same as before, join ST cutting



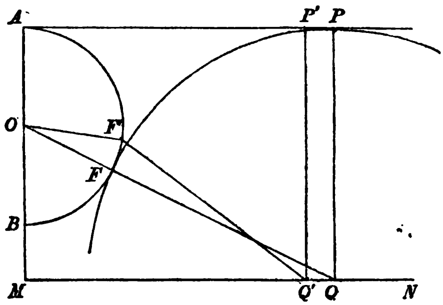
$MF'F''$ in F''. Draw $F''P'N'$ perpendicular to MS cutting OS in P'. Find F' so that $P'F'=P'F''=P'N'$. P' is a point in the trajectory whose focus is F'. Hence the tangent

at P' bisects $F'P'N'$. But OP' bisects $F'P'F''$. Hence the trajectory at P' is perpendicular to OS.

Also as F'' is the focus of the other path by which the point P' might be reached, P' will be the vertex of that path, and therefore the particle will be moving horizontally when it reaches P'.

117. Even if the plane along which the range is measured do not pass through the point of projection, a somewhat similar construction will enable us to find the direction of projection for the maximum range. Thus,

Let it be required to find the direction of projection from

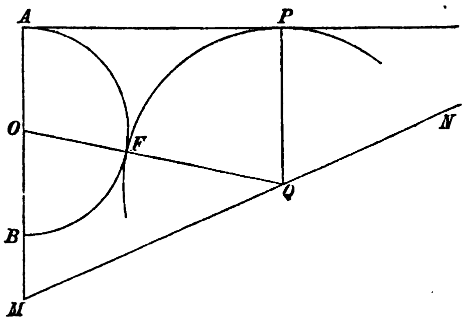


O with velocity due to AO in order that the range on a horizontal line MN may be a maximum.

Suppose Q' the point where the projectile falls. Join $Q'F'$, $F'O$, F' being the focus of the path. Then if $Q'P'$ be vertical and meet the horizontal line through A in P', we have $F'Q' = Q'P'$. This is true of each of the paths, and $Q'P'$ is constant. The farthest point Q which can be reached will therefore be determined by inflecting OQ to MN, where $OQ = OA + PQ$, and therefore if $AO = a$, $AM = b$ the cosine of double the requisite angle of elevation will be $\left(\dfrac{b-a}{b+a}\right)$.

Should MN be an inclined plane, we must evidently draw a line QO, and the corresponding vertical QP; such that if QO meet the circle in F, $FQ = QP$.

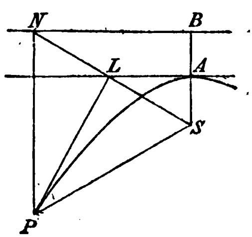
This resolves itself into the well-known geometrical problem of describing a circle whose centre is in a given line, and which touches a given circle, and a given straight line.



Of the two solutions, which this problem admits of, one belongs to MN, the other to MN produced to the other side of the point of projection.

118. Perhaps, however, the most satisfactory method of solving all such problems about the maximum range, is to describe the parabola which envelops all the trajectories. The point where this cuts the plane, &c. on which the range is estimated, gives the maximum value of the range, and it is then easy from known properties of the envelop to construct for the required path.

119. Let P be any point in the trajectory, S its focus, BN, AL, the directrix, and the tangent at the vertex.



Then (velocity at $P)^2 = 2g\,PN = 2g\,SP$

$= $ (by a property of the parabola) $\dfrac{2g}{SA}\,SL^2 = \dfrac{g}{2SA}\,SN^2$.

Hence velocity at $P \propto SN$; and, since by the figure $SL = LN$, PL is the tangent at P and is perpendicular to SN.

Hence as SN is perpendicular to the direction of motion at P, proportional to the velocity at P, and drawn from a fixed point S, the locus of N is the Hodograph (§ 20) turned through a right angle about S. As this is a horizontal straight line, the Hodograph is a vertical line.

This result will be found of considerable utility in solving various problems in the common vacuum theory of projectiles. It is evident that SB, BN represent the horizontal and vertical velocities at P, on the same scale on which SN represents the entire velocity at that point.

120. It may be interesting to anticipate a little here, by introducing matter properly belonging to the next chapter. We wish to shew that the above geometrical constructions can easily be extended to paths of projectiles when they are so large as to require us to take account of the variations in the *direction* and *amount* of gravity. The following sections are taken from the *Proc. R. S. E.* 1865-6.

121. When, instead of supposing gravity to be of constant amount, and to act in parallel lines, we take the more accurate assumption that it tends to the centre of the Earth, and varies inversely as the square of the distance from that point, Chapter V. shews us that in general the path of the projectile is an *Ellipse*, one of whose foci is at the Earth's centre, and the length of whose major axis depends only on the *velocity* of projection. The following propositions (among many others analogous to those just given) may then be enuntiated.

1. The locus of the second foci of the paths of all projectiles leaving a given point, with a given velocity, in a vertical plane, is a circle.

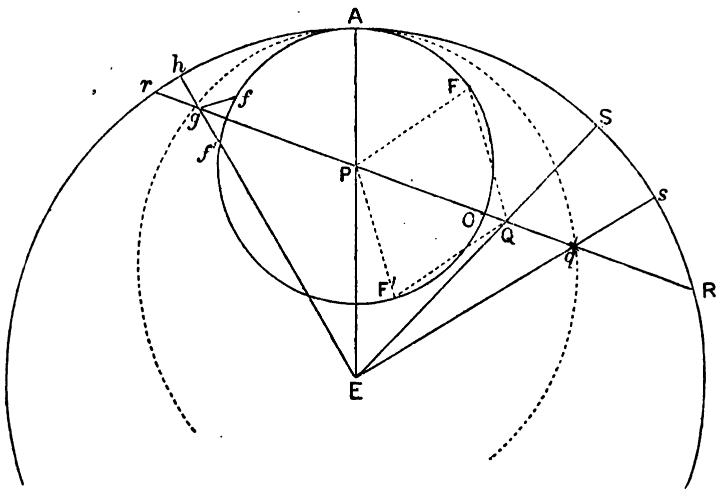
2. The direction of projection, for the greatest range on a given line, passing through the point of projection, bisects the angle between the vertical and the line.

3. Any other point on the line, which can be reached at all, can be reached by *two* different paths, and the directions of projection for these are equally inclined to the direction which gives the maximum range.

4. If a projectile meet the line at right angles, the point which it strikes is the vertex of the other path by which it may be reached.

5. The envelop of all possible paths in a vertical plane is an ellipse, one of whose foci is the centre of the earth, and the other the point of projection.

The proofs of these propositions are extremely simple. Thus, let E be the earth's centre, P the point of projection,



A the point which the projectile would reach if fired vertically upwards. With centre E, and radius EA, describe a circle in

the common plane of projection. This, the circle of zero velocity, corresponds to the common directrix of the parabolic paths in the ordinary theory. If F be the second focus of any path, we must have $EP+PF$ constant, because the axis major depends on the *velocity*, not the *direction*, of projection. Hence (1) the locus of F is the circle AFO. Again, since, if F be the focus of the path which meets PR in Q, we must have $FQ = QS$, it is obvious that the greatest range Pq is to be found by the condition $Oq = qs$. O is therefore the second focus of this trajectory, and therefore (2) the direction of projection for the greatest range on PR bisects the angle APR. If $QF = QF' = QS$, F and F' are the second foci of the two paths by which Q may be reached; and, as $\angle FPO = \angle F'PO$, we see the truth of (3). If Q be a point reached by the projectile when moving in a direction perpendicular to PR, we must evidently have $PQF' = \angle PQF = \angle SQR = \angle EQP$; i.e. EQ passes through F'. This case is represented on the other side of the diagram —where $f'g = gh = fg$. The ellipse whose second focus is f evidently meets Pr at right angles: and that whose second focus is f' has (4) its vertex at g. The locus of q is evidently the envelop of all the trajectories. Now

$$Pq = PO + Oq = PA + Oq,$$

$$Eq = Es - sq = EA - Oq.$$

Hence

$$Pq + Eq = PA + AE,$$

or (5) the envelop is an ellipse, whose foci are E and P, and which passes through A.

122. *When a particle moves subject to the action of two centres, one attractive and the other repulsive, where the law is the direct distance and the strengths the same, its motion will be the same as that of a projectile in vacuo.*

For the whole force on the particle resolved perpendicular to the line joining the centres is evidently zero, and that parallel to this line is equal to that which would be exerted by either of the centres on a particle placed at the other; and

always tends in the direction parallel to that from the repelling, to the attracting, centre. It corresponds therefore exactly to gravity, within moderate elevations above the earth's surface.

123. Again, if a particle moves on a plane inclined to the horizon at an angle θ, the acceleration is, by § 84, $g\sin\theta$ parallel to the line of greatest slope on the plane, and therefore the trajectory will still be a parabola, whose dimensions will depend upon θ.

Ex. *A particle is projected from a given point with a given velocity, and moves on an inclined plane; find the locus of the directrices of its path for different inclinations of the plane.*

It will be easily seen that when a particle moves on an inclined plane, the velocity at any point is equal to that which would have been acquired by sliding from the directrix; that is (§ 85) equal to the velocity due to the fall from a horizontal plane through the directrix. Now the velocity is given constant, hence the locus of the directrices is a horizontal plane.

124. *A particle moves subject to an attraction always perpendicular to a given plane, its intensity being a function of the distance of the particle from the plane: to determine the motion.*

It is evident that the motion will be confined entirely to a plane through the direction of projection perpendicular to the attracting plane. Let us take the plane of motion as that of xy, the axis of x lying in the attracting plane. Let $\phi'(D)$ be the acceleration at distance D, where ϕ' is the derived function of ϕ. Then the equations of motion are

$$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=-\phi'(y).$$

Suppose the particle projected from a point (a, b), in a direction making an angle α with the axis of x, and with a velocity V.

Multiplying by $\frac{dx}{dt}$, $\frac{dy}{dt}$, and integrating we get

$$\left.\begin{aligned}\frac{1}{2}\left(\frac{dx}{dt}\right)^2 &= \text{const.} = \frac{1}{2}V^2\cos^2\alpha,\\ \frac{1}{2}\left(\frac{dy}{dt}\right)^2 &= C - \phi(y) = \frac{1}{2}V^2\sin^2\alpha + \phi(b) - \phi(y).\end{aligned}\right\}\ldots\ldots(1).$$

Hence

$$\frac{1}{2}v^2 = \frac{1}{2}V^2 + \phi(b) - \phi(y),$$

or

$$\frac{1}{2}v^2 + \phi(y) = \frac{1}{2}V^2 + \phi(b),$$

a particular case of conservation of energy.

To find the differential equation of the path, we have

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{\sqrt{[V^2\sin^2\alpha + 2\{\phi(b) - \phi(y)\}]}}{V\cos\alpha},$$

an equation integrable for particular forms only of the function ϕ.

An interesting case is that in which the attraction of the plane is inversely as the cube of the distance,

or $\qquad \phi'(y) = \frac{\mu}{y^3}$, and therefore $\phi(y) = -\frac{1}{2}\frac{\mu}{y^2}$.

The differential equation becomes

$$\frac{dy}{dx} = \frac{\sqrt{\left(\frac{\mu}{y^2} + V^2\sin^2\alpha - \frac{\mu}{b^2}\right)}}{V\cos\alpha}$$

or

$$\frac{dx}{V\cos\alpha} = \frac{ydy}{\sqrt{\left\{\mu + \left(V^2\sin^2\alpha - \frac{\mu}{b^2}\right)y^2\right\}}},$$

and integrating

$$\frac{x-a}{V\cos\alpha}$$

$$=\frac{\sqrt{\left\{\mu+\left(V^2\sin^2\alpha-\frac{\mu}{b^2}\right)y^2\right\}}-\sqrt{\left\{\mu+\left(V^2\sin^2\alpha-\frac{\mu}{b^2}\right)b^2\right\}}}{V^2\sin^2\alpha-\frac{\mu}{b^2}},$$

the equation of a conic; an ellipse, parabola, or hyperbola, according as

$$V^2\sin^2\alpha-\frac{\mu}{b^2}$$

is negative, zero, or positive.

We might have obtained the above results by integrating separately the two equations of motion, and then eliminating t between them.

For a repulsion, instead of an attraction, it is easy to see, by a slight modification of the above process, that there is only one case, and that the curve described is a hyperbola whose *conjugate* axis lies in the intersection of the plane of projection and the attracting plane.

From this we see that the conic sections are the only curves which can be described by a free particle moving in a plane with acceleration in the direction, and inversely as the cube, of the perpendicular distance from a given line in that plane.

The converse of either of the above propositions is easily investigated; thus, taking the first, our problem becomes

125. *To find the attraction perpendicular to an axis that a free particle may describe a conic section.*

Take the axis as that of x, and the vertex as origin, then the equation

$$y^2=2mx+nx^2\dots\dots\dots\dots\dots\dots(1)$$

will represent, by properly taking m and n, any parabola, any hyperbola referred to its transverse axis, or any ellipse referred to either axis.

Since the attraction is perpendicular to the axis, we have

$$\frac{dx}{dt} = c.$$

Hence $$y\frac{dy}{dt} = mc + nxc;$$

and $$y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = nc^2.$$

From these $$\frac{d^2y}{dt^2} = \frac{1}{y}\left\{nc^2 - \left(\frac{dy}{dt}\right)^2\right\}$$

$$= \frac{1}{y}\left\{nc^2 - \frac{(m+nx)^2}{y^2}c^2\right\}$$

$$= \frac{c^2}{y^3}(ny^2 - m^2 - 2mnx - n^2x^2)$$

$$= -\frac{c^2m^2}{y^3} \text{ by equation (1).}$$

For a hyperbola with its conjugate axis in the axis of x, the equation is

$$y^2 = p^2x^2 + q^2.$$

Hence $$y\frac{dy}{dt} = p^2x\frac{dx}{dt}$$

$$= p^2cx,$$

from which we have immediately

$$y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 = p^2c^2.$$

That is, $$\frac{d^2y}{dt^2}=\frac{1}{y}\left\{p^2c^2-\left(\frac{dy}{dt}\right)^2\right\}$$

$$=\frac{p^2c^2}{y}\left(1-\frac{p^2x^2}{y^2}\right)=\frac{p^2q^2c^2}{y^3},$$

which indicates a *repulsion* inversely as the cube of the distance from the conjugate axis.

126. *To find the repulsion which must act perpendicular from a plane, in terms of the distance from that plane, that a given path may be described.*

Take the axes as before; then, Y being the acceleration due to the repulsion (a function of y only), we have

$$\frac{d^2x}{dt^2}=0, \text{ or } \frac{dx}{dt}=\text{const.}=a, \text{ suppose};$$

$$\frac{d^2y}{dt^2}=Y \quad\text{.....................(1).}$$

Let $y=f(x)$ be the equation of the given curve, then

$$\frac{dy}{dt}=af'(x),$$

$$\frac{d^2y}{dt^2}=a^2f''(x),$$

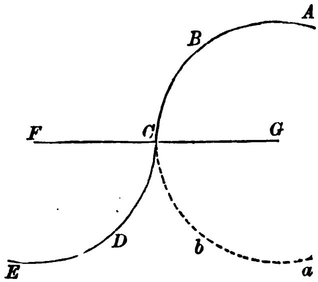
or by (1), $$Y=a^2f''(x)$$

$$=a^2f''\{f^{-1}(y)\},$$

by the equation of the curve. Hence, as f is a given function, the acceleration and the repulsion are found.

127. It is necessary to observe that, in the case of § 124, when the particle actually reaches the axis, it will not proceed to describe the portion of the same curve which lies on the

other side of the axis, as this would involve a change in sign of the constant horizontal velocity. It is, in fact, evident that



in such cases the particle having described ABC will, instead of pursuing the course Cba, actually describe CDE similar and equal to Cba, but turned in the opposite direction. And a similar remark applies to the general problem in § 126.

Although, in the case of ABC being a conic, one of whose axes is CG, and therefore cutting it at right angles in C, it might seem that at C the horizontal velocity vanishes, yet it is to be recollected that the velocity at C is infinitely great; and it may easily be shewn by independent methods, such as the method of limits, if the foregoing analysis do not appear satisfactory, that the velocity parallel to CG is really constant throughout the motion.

128. It may be useful to notice that cases of this kind are reduced at once to investigations similar to those of the last Chapter, by considering, separately, the equations of motion parallel and perpendicular to the attracting plane.

Whenever, then, we can completely determine the motion of a particle in a straight line towards a centre, we can also completely solve the problem of the motion of a particle anyhow projected, and attracted by an infinite plane; the intensity in terms of the distance being the same in the two cases.

129. *Generally, when a particle is anyhow projected and subject only to an acceleration whose direction is perpendicular to a given plane, and whose magnitude depends solely on the distance from the plane; the velocity parallel to that plane is constant; and, in passing from any point to another, the square of the velocity is altered by a quantity depending only upon the distances of those two points from the given plane.*

Take the axis of y perpendicular to the given plane, and the axis of x in it, so that the direction of projection lies in xy. This will evidently be the plane of motion; and the equations are

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = Y = -\phi'(y) \text{ suppose.}$$

Hence
$$\frac{dx}{dt} = c,$$

and
$$\frac{1}{2}v^2 = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\left(\frac{dy}{dt}\right)^2 = \frac{1}{2}V^2 + \int_{y_1}^{y} Ydy$$
$$= \frac{1}{2}V^2 + \phi(y_1) - \phi(y),$$

or
$$\frac{1}{2}v^2 + \phi(y) = \frac{1}{2}V^2 + \phi(y_1),$$

V being the velocity of projection, and y_1 the co-ordinate of the point of projection; which proves the proposition.

This is, of course, merely a particular case of the general principle of Conservation of Energy (§ 78); $\phi(y)$ being the Potential, and $\frac{1}{2}v^2$ the Kinetic, Energy of unit mass.

130. As another example of the motion of a particle under the action of forces whose direction is constant, *let us consider the motion of a particle of light in the corpuscular*

theory, at the confines of two homogeneous isotropic media whose bounding surface is plane.

In this theory the hypothesis is that the attractions or repulsions, exerted by the particles of any medium on a particle of light passing through it, are insensible at sensible distances but enormously great at infinitely small distances. Hence of course the path of such a particle in a homogeneous medium will be a straight line, and will be described with constant velocity, until the particle is infinitely near to the bounding surface of the medium.

Thus, suppose AB to be the common plane surface of two such media. Draw CD at a distance from AB equal to that at which the intensity of the attractions of the particles of the medium begins to be sensible; and draw EF parallel to CD and equidistant from it with AB. By what we have just noticed, a particle of light moving along PQ will arrive at Q without any change of velocity or direction. Also from the symmetry of the figure, the resultant of all the sensible attractions or repulsions on it will always be perpendicular to AB. This shews, § 129, that the velocity resolved parallel to AB is constant throughout the motion, and also that whatever be the direction of PQ, the change in the square of the velocity in passing from Q to any point of the path will depend only on the distance of that point from AB.

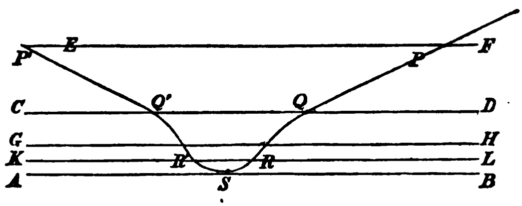
Let PQR represent a portion of the path.

We have no means of determining its actual form, since the extent through which the attraction is sensible, the law of its variation, and whether it change from attraction to repulsion with the distance, are unknown.

Through any point R draw KRL parallel to AB, and let GH be equidistant from KL with AB.

Then at R the particle is subject only to the actions of the upper medium beyond GH, and of the lower medium.

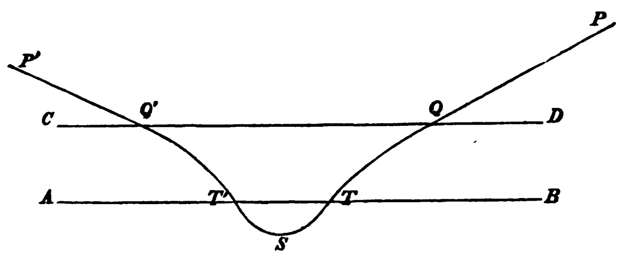
If the resultant effects of these two should, at a point S in the superior medium, destroy the velocity perpendicular to AB, the particle will evidently pursue a course $SR'Q'P'$



similar and equal to $SRQP$, and the angles $P'Q'C$ and PQD will be equal, as also the velocities in PQ and $P'Q'$. (§ 129.) *Here we have the case of a ray reflected at a plane surface.*

If, however, the attraction of the lower medium should so prevail that the particle actually enters it, then we may consider its motion, while it is still within the range of action of both media, precisely as before; but there will be two cases.

I. At some point as S whose distance from AB (the bounding surface) is less than that of AB from CD, the velocity perpendicular to AB may be destroyed; then, as before,



the particle will pursue the path $ST'Q'P'$, similar and equal to $STQP$, and will be reflected at an angle equal to that of incidence and with its original velocity.

II. The particle may pass into the lower medium so far as to be independent of the action of the upper medium. After this it will move in a straight line as before, and the change of the square of its velocity will be, § 129, independent

of the path pursued. Hence, if V be the velocity, and α the angle, of incidence; V', α' those of refraction, we have, by the condition that the velocity parallel to the surface is unaltered,

$$V \sin \alpha = V' \sin \alpha'.$$

Also by the fixed amount of change in the square of the whole velocity,

$$V^2 = V'^2 \pm a^2,$$

where a is a constant depending on the nature of the two media.

Hence, $$\frac{\sin \alpha}{\sin \alpha'} = \frac{V'}{V} = \sqrt{\left(1 \mp \frac{a^2}{V^2}\right)},$$

and, therefore, for particles of light which have the same velocity the ratio of the sines of the angles of incidence and refraction is constant. *This is the known law of ordinary refraction.* Unfortunately, however, in order that a ray may be bent, at refraction, *towards* the normal to the refracting surface (i.e. so that $\alpha' < \alpha$) we must have $V' > V$; a result lately shewn to be inconsistent with experiment.

We have introduced this example, although belonging to a theory now completely exploded, as it forms a good illustration of the application of the results of this Chapter, and afforded the first instance of the solution of a problem connected with molecular actions. It is due to Newton.

EXAMPLES.

(1) The time of describing any portion PQ of the parabolic path of a particle under gravity, is proportional to the difference of the tangents of the angles which the tangents at P and Q make with the horizon. (§ 119.)

(2) If a shell burst, all the fragments receiving equal velocities from the explosion; shew that the locus of the foci of the paths of the fragments is a sphere, of the vertices an oblate spheroid, and that the particles themselves at any instant will lie on a sphere.

(3) Two bodies, projected from the same point A, in directions making angles α, α' with the vertical, pass through the point B in the horizontal plane through A; prove that if t, t' be the times of flight from A to B,

$$\frac{\sin(\alpha-\alpha')}{\sin(\alpha+\alpha')}=\frac{t'^2-t^2}{t'^2+t^2}.$$

(4) If u and v be the velocities at the ends of a focal chord of a projectile's path, V_x the horizontal velocity, shew that

$$\frac{1}{u^2}+\frac{1}{v^2}=\frac{1}{V_x^2}. \quad (\S\ 119.)$$

(5) From a point in an inclined plane two bodies are projected with the same velocity in the same vertical plane in directions at right angles to each other; prove that the difference of their ranges is constant.

(6) If v, v', v'', be the velocities at three points P, Q, R, of the path of a projectile where the inclinations to the horizon are $\alpha-\beta$, α, $\alpha+\beta$; and if t, t' be the times of describing PQ, QR respectively, shew that

$$v''t=vt', \text{ and } \frac{1}{v}+\frac{1}{v''}=\frac{2\,.\cos\beta}{v'}. \quad (\S\ 119.)$$

(7) If two particles be projected from the same point at the same instant in the same vertical plane, with velocities v and v_1 in directions making angles α and α_1 with the horizon; shew that the interval between their transits through the other point which is common to their paths is

$$\frac{2}{g}\cdot\frac{vv_1\sin(\alpha\sim\alpha_1)}{v_1\cos\alpha_1+v\cos\alpha}.$$

(8) Particles slide from rest at the highest point of a vertical circle down chords, and are then allowed to move freely; shew that the locus of the foci of their paths is a circle of half the radius, and that all the paths bisect the vertical radius.

(9) If the particles slide down chords to the lowest point, and be then suffered to move freely, the locus of the foci is a cardioid.

(10) Down what chord from the vertex of a vertical circle must a particle slide so as to have when falling freely the greatest range on a given horizontal plane?

(11) Find the locus of the foci of all trajectories which pass through two given points.

(12) Particles fall down *diameters* of a vertical circle; the locus of the foci of their subsequent paths is the circle.

(13) If a body describe a cycloid under an attraction to the axis, shew that the attraction varies inversely as $2\sin\theta - \sin 2\theta$, θ being the corresponding arc of the generating circle measured from the vertex.

(14) If the acceleration be perpendicular to a plane and vary as the distance, shew that the curves described have equations of the form

$$\left.\begin{aligned} y &= Aa^x + Ba^{-x}, \\ \text{or } y &= A\cos(mx + B) \end{aligned}\right\} \text{ for a repulsion or attraction respectively.}$$

Find the circumstances of projection in the two cases that the curves may be the catenary, and the companion to the cycloid, respectively.

(15) Particles are projected in the same plane and from the same point, in such a manner that the parabolas described are equal; prove that the locus of the vertices of these parabolas will be a parabola.

(16) Find the direction of projection, with a given velocity, from a given point, so that a given plane, not passing through the point, may be reached in the least possible time.

(17) Particles slide down radii vectores of the curve whose equation is $r = f(\theta)$, the plane of the curve being

vertical and θ being measured from a horizontal line, prove that the locus of the foci of their future paths is the curve

$$r = \cos\frac{\theta}{2} f\left(\frac{\theta}{2}\right).$$

(18) Through a point an inclined plane is drawn, and from that point a particle is projected with a given velocity so that its direction of motion when it meets the plane again cuts it at right angles; shew that the locus of the point of meeting for different positions of the inclined plane is an ellipse.

(19) The attraction between two particles is $\frac{\mu m^2}{r^3}$, where m is the mass of each particle, and r the distance between them, and they are projected with equal velocities on the same side of the line (c) joining them in directions not parallel but equally inclined to that line; prove that the path of each will be an ellipse, parabola, or hyperbola, according as the initial component of each velocity in direction of the line (c) is less than, equal to, or greater than $\sqrt{\frac{2\mu m}{c^2}}$.

(20) A perfectly elastic particle is projected so as to strike on the inside a surface of revolution of which the axis is vertical and given in position. Shew that the vertices of all the parabolic orbits described after successive rebounds lie on a surface which is independent of the surface of revolution.

(21) If α be the angle of elevation required in order that a bullet may have a certain range on a horizontal plane, θ the additional elevation required above a plane inclined to the horizon at an angle β,

$$\tan\theta = \frac{\sin\beta \sin^2\alpha}{\sin(2\alpha+\beta)}.$$

(22) A particle is projected from a given point with a given velocity u so that the range on a given inclined plane may be the greatest possible: prove that, if v be its final

velocity, and a perpendicular be let fall on the given plane from the point of intersection of the initial and final directions of motion, the length of the perpendicular is $\frac{uv}{2g}$.

(23) A cycloidal arc is placed with its axis vertical and vertex upwards, and a particle is projected so as, after moving in contact with the arc for a finite distance, to describe a parabola freely; prove that the focus of the parabola lies on a cycloid of half the dimensions having the same base.

(24) Shew that the whole area commanded by a gun on a hill-side is an ellipse whose focus is at the gun, whose excentricity is the sine of the inclination of the hill to the horizon, and whose semi-latus-rectum is the greatest height to which the gun could send a ball.

CHAPTER V.

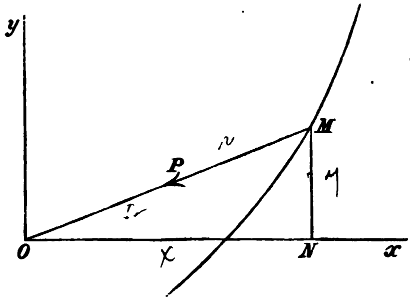
CENTRAL ORBITS.

131. IN this part of the subject we consider the motion of a particle under the action of an attraction or repulsion whose direction always passes through, and whose intensity is some function of the distance from, a fixed point. The fixed point is called the *Centre*. The case of attraction, as including the most important applications of the subject, we will take as our standard case; but it will be seen that a simple change of sign will adapt our general formulæ to repulsion. If the centre of attraction be itself in motion, the methods of §§ 26, 31, enable us easily to treat it as fixed; but in this case the relative acceleration is not in general directed to the centre, so that the problem no longer belongs to *Central Orbits* strictly so called. It will be considered later. If the centre be moving with constant velocity in a straight line, the results of this chapter are at once applicable to the relative motion.

132. *A particle is projected in a plane, and is acted on by an attraction* P *directed to the fixed point* O *in that plane; to determine the motion.*

The whole motion will clearly take place in the plane, as there is nothing to withdraw the particle from it. Let Ox, Oy, any two lines through O at right angles to each other, be taken as the axes of co-ordinates. Let M be the position of the particle at the time t; and draw MN perpendicular to Ox, and join MO. Let $ON=x$, $NM=y$, $OM=r$, and the angle $NOM=\theta$. Then, since $\cos\theta=\frac{x}{r}$, $\sin\theta=\frac{y}{r}$, the com-

ponents of P, parallel to the axes, are $-P\frac{x}{r}, -P\frac{y}{r}$. But by the second law of motion we may consider the accelerations



in the directions of x and y separately, and we have therefore

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -P\frac{x}{r}\\ \frac{d^2y}{dt^2} &= -P\frac{y}{r}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(A).$$

In these, since P is a function of r, and therefore of x and y, the second members will generally contain both these variables, and the equations must be treated as simultaneous differential equations. Their integrals will give $x, y, \frac{dx}{dt}, \frac{dy}{dt}$, in terms of t; from which the position and velocity of the particle at any instant will be known, and the problem completely solved. In one case, however, viz. when P is proportional to r, the first equation will involve x and t, and the second y and t, only, and each equation may be integrated by itself. As it is the simplest example of its class, and of great importance in its applications, especially to Acoustics and to Physical Optics, we will begin by considering it.

133. *A particle moves about a centre of attraction varying directly as the distance: to determine the motion.*

Let μ be the acceleration at unit of distance, called the

strength of the centre, then $P=\mu r$, and equations (A) become

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=-\mu x\\ \frac{d^2y}{dt^2}&=-\mu y\end{aligned}\right\}\text{..................}(B),$$

the integrals of which, see § 83, are

$$x=A\cos\{\sqrt{\mu}t+B\}\text{.....................}(1),$$

$$y=A'\cos\{\sqrt{\mu}t+B'\}\text{.....................}(2),$$

A, B, A', B' being the constants introduced in the integration, to be determined by the initial circumstances of motion. Consider the particle projected from a point on the axis of x, at distance a from the centre, with velocity V, and in a direction making an angle α with Ox. When $t=0$, we have $x=a$, $y=0$, $\frac{dx}{dt}=V\cos\alpha$, $\frac{dy}{dt}=V\sin\alpha$. Hence,

$$\begin{aligned}a&=A\cos B,\\ 0&=A'\cos B',\\ V\cos\alpha&=-A\sqrt{\mu}\sin B,\\ V\sin\alpha&=-A'\sqrt{\mu}\sin B'.\end{aligned}$$

Expanding the cosines in (1) and (2), and substituting these expressions for the constants, we obtain

$$x=\frac{V\cos\alpha}{\sqrt{\mu}}\sin\sqrt{\mu}t+a\cos\sqrt{\mu}t\text{ ....}(3),$$

$$y=\frac{V\sin\alpha}{\sqrt{\mu}}\sin\sqrt{\mu}t\text{ ....}(4),$$

which contain the complete solution of the problem. Eliminating t, we have

$$(x\sin\alpha-y\cos\alpha)^2+\frac{\mu a^2}{V^2}y^2=a^2\sin^2\alpha,$$

the equation of the path of the particle; which is therefore an ellipse whose centre is O. Equations (3) and (4) give periodic values for x, y, $\frac{dx}{dt}$, $\frac{dy}{dt}$, such that all the circumstances of motion will be the same at the time $t+\frac{2\pi}{\sqrt{\mu}}$ as at the time t. The period of revolution is therefore $\frac{2\pi}{\sqrt{\mu}}$: a most remarkable result, as it is independent of the dimensions of the ellipse, and depends solely on the intensity of the force.

By taking μ negative in equations (B), we may apply them to the case of a repulsion varying as the distance from O. In the integration for this supposition the sines and cosines would be replaced by exponentials, and the curve described would be a hyperbola having O as centre; but the motion would not be one of revolution, as the particle would necessarily always remain on the same branch of the hyperbola.

134. Recurring to equations (A), it will in all cases but the one we have just considered be more convenient to transform them to polar co-ordinates, especially as the general polar differential equation of the orbit described by a particle under the action of a central force can be easily formed, as follows.

135. *A particle being acted on by a central attraction; it is required to determine the polar equation of the path.*

Multiplying the second of equations (A), § 132, by x, and the first by y, and subtracting, we obtain

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=0.$$

Integrating,

$$x\frac{dy}{dt}-y\frac{dx}{dt}=\text{constant}=h\text{ suppose.}$$

Changing the variables from x, y, to r, θ, where $x=r\cos\theta$, $y=r\sin\theta$, we get, as in § 24,

$$r^2\frac{d\theta}{dt}=h \qquad\qquad (1),$$

or, substituting $\frac{1}{u}$ for r,

$$\frac{d\theta}{dt}=hu^2 \qquad\qquad (2).$$

Again, $\qquad x=r\cos\theta=\frac{\cos\theta}{u}$;

which gives $\qquad \frac{dx}{dt}=-\frac{u\sin\theta+\cos\theta\frac{du}{d\theta}}{u^2}\frac{d\theta}{dt}$

$$=-h\left(u\sin\theta+\cos\theta\frac{du}{d\theta}\right), \text{ by (2)};$$

and therefore $\frac{d^2x}{dt^2}=-h\left(u\cos\theta+\cos\theta\frac{d^2u}{d\theta^2}\right)\frac{d\theta}{dt}$

$$=-h^2u^2\left(u\cos\theta+\cos\theta\frac{d^2u}{d\theta^2}\right), \text{ by (2)}.$$

But, by the first of equations (A),

$$\frac{d^2x}{dt^2}=-P\cos\theta.$$

Equating these values of $\frac{d^2x}{dt^2}$, and dividing by $\cos\theta$, we have

$$P=h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) \qquad\qquad (3),$$

or

$$\frac{d^2u}{d\theta^2}+u-\frac{P}{h^2u^2}=0 \qquad\qquad (4).$$

This is the differential equation of the orbit described; and as, in any particular instance, P will be given in terms of r, and therefore in terms of u, its integral will be the polar equation of the required path.

136. It may easily be obtained by the formulæ of § 16, and, as this method is instructive as well as useful, we give it for the case, when in addition to the central acceleration due to the attraction P there is a transverse acceleration T impressed on the particle.

Instead of equations (A) we may evidently write (by §§ 16, 69),

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P,$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = T.$$

Putting $r^2\frac{d\theta}{dt} = h$, and $u = \frac{1}{r}$, then $\frac{d\theta}{dt} = hu^2$; and the second equation becomes

$$\frac{dh}{dt} = \frac{T}{u},$$

or

$$\frac{dh^2}{d\theta} = \frac{2T}{u^3}.$$

Also

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta},$$

$$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2} - T\frac{1}{u}\frac{du}{d\theta},$$

and

$$r\left(\frac{d\theta}{dt}\right)^2 = h^2u^3.$$

Therefore

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 - T\frac{1}{u}\frac{du}{d\theta} = -P,$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} - \frac{T}{h^2u^3}\frac{du}{d\theta}.$$

137. The general integrals of (A), which are differential equations of the second order, ought to contain four constants. One of these has been already introduced in (1), and two more will be introduced by the integration of (4). If the value of r in terms of θ deduced from the integral of (4) be

substituted in (1), and that equation be then integrated, the remaining constant will be introduced, and the path of the particle and its position at any time will be obtained. The four constants involved in the resulting equations must be determined from the initial circumstances of motion; namely, the initial position of the particle (depending on *two* independent co-ordinates), its initial velocity, and its direction of projection.

138. Equation (3) may be used to ascertain the law of central attraction which must act upon a particle to cause it to describe a given curve. To effect this we must determine the relation between u and θ from the polar equation of the proposed orbit referred to the required centre as pole: we must then differentiate u twice with respect to θ, and substitute the result in the expression for P; eliminating θ, if it be involved, by means of the relation between u and θ. In this way we shall obtain P in terms of u alone, and therefore of r alone.

When we know the relation between r and θ, from (4), we make use of equation (1) to determine the time of describing a given portion of the orbit; or, conversely, to find the position of the particle in its orbit at any time.

139. The equation of the orbit between r and p, the radius vector and the perpendicular on the tangent at any point, may be easily obtained from (4). For by *Diff. Calc.* we have

$$\frac{d^2u}{d\theta^2}+u=\frac{1}{p^3u^2}\frac{dp}{dr};$$

and therefore
$$P=\frac{h^2}{p^3}\frac{dp}{dr}.$$

p44 **140.** *The sectorial area swept out by the radius vector of the particle in any time is proportional to the time* (§ 24).

If A denote this area, we have, by *Diff. Calc.*,

$$\frac{dA}{dt}=\frac{1}{2}r^2\frac{d\theta}{dt};$$

and therefore, by equation (1) of § 135,

$$\frac{dA}{dt} = \frac{1}{2}h,$$

whence $$A = \tfrac{1}{2}ht,$$

if A and t be supposed to vanish together.

Therefore the areas described in different intervals are proportional to these intervals.

We also see, by taking $t = 1$, that the value of h is twice the area described in a unit of time.

141. *The velocity of the particle at each point of its path is inversely proportional to the perpendicular from the centre on the tangent at that point.* (§ 23.)

For $$\text{Velocity} = v = \frac{ds}{dt}$$

$$= \frac{ds}{d\theta}\frac{d\theta}{dt}$$

$$= \frac{r^2}{p}\frac{d\theta}{dt}, \text{ by } \textit{Diff. Calc.}$$

(p being the perpendicular on the tangent from the centre)

$$= \frac{h}{p}, \text{ by equation (1) of § 135.}$$

Hence, as above, $$v \propto \frac{1}{p}.$$

142. This equation enables us to express h in terms of the initial circumstances of the motion. For, let R be the distance of the point of projection from the centre, V the velocity, and β the angle which the direction of projection makes with that of R. Then evidently the perpendicular on tangent at point of projection $= R \sin \beta$;

$$\text{or } V = \frac{h}{R \sin \beta},$$

whence $$h = VR \sin \beta.$$

Again, since by *Diff. Calc.*,

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2,$$

we have

$$v^2 = \frac{h^2}{p^2} = h^2 \left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\},$$

another important expression for the velocity.

143. *The velocity at any point of a central orbit is independent of the path described, and depends solely on the intensity of the attraction, the distance of the point from the centre, and the velocity and distance of projection.*

Multiply equations (A) § 132, by $\frac{dx}{dt}, \frac{dy}{dt}$ respectively, and add, then

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} = -\frac{P}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= -P\frac{dr}{dt}.$$

$$\left(\text{Since } x^2 + y^2 = r^2, \text{ we have } x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}\right).$$

But $v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$;

hence $$\frac{1}{2}\frac{d(v^2)}{dt} = -P\frac{dr}{dt}.$$

Also, since P is a function of r alone, let $P = \phi(r)$, then

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}V^2 = -\int_R^r \phi(r)\,dr$$

$$= \phi_1(R) - \phi_1(r),$$

if at the point of projection $v = V$, $r = R$.

If the velocity vanishes at a distance a from the centre,

$$\tfrac{1}{2}v^2 = \phi_1(a) - \phi_1(r)$$

and a is called the radius of the circle of zero velocity. (Compare § 78.)

144. *The velocity of a particle at any point of a central orbit is the same as that which would be acquired by a particle moving freely from rest along one-fourth of the chord of curvature at the point, drawn through the centre, under the action of a constant force whose intensity is equal to that of the central attraction at the point.*

By § 143,

$$\frac{1}{2}\frac{d(v^2)}{dt} = -P\frac{dr}{dt};$$

or
$$v\frac{dv}{dr} = -P.$$

And by § 141,

$$v = \frac{h}{p}.$$

Differentiating the logarithm of the latter, we obtain

$$\frac{1}{v}\frac{dv}{dr} = -\frac{1}{p}\frac{dp}{dr},$$

and, dividing the former equation by this,

$$\tfrac{1}{2}v^2 = \tfrac{1}{2}Pp\frac{dr}{dp} = P\frac{p}{r}\frac{2r}{4}\frac{dr}{dp}$$

$$= P\tfrac{1}{4}q,$$

where q is the chord of curvature through the centre. Hence the proposition, § 82.

From this it follows that the velocity, V, of a particle

moving in a circle of radius R, under the action of an attraction P to the centre, is given by the equation

$$V^2 = PR,$$

a simple, and most useful expression*.

145. Def. An *Apse* is a point in a central orbit at which the radius vector is a maximum or minimum, and the corresponding value of the radius vector is called an *Apsidal Distance.*

The analytical conditions for such a point are that $\frac{du}{d\theta}$ should vanish, and that the first succeeding differential coefficient which does not vanish should be of an even order. The first condition ensures that the tangent at an apse is perpendicular to the radius vector.

* The results of the last few Articles may be obtained in the following manner.

By §§ 49 and 65

$\frac{d^2s}{dt^2}$ = Resolved part of P along the tangent of the orbit $= -P\frac{dr}{ds}$ ....(1),

$\frac{v^2}{\rho}$ = Resolved part of P along the normal $= P\frac{p}{r}$ ....(2).

Multiply (1) by $\frac{ds}{dt}$ and integrate, then

$$\frac{1}{2}\left(\frac{ds}{dt}\right)^2 = -\int P dr.$$

From (2)

$$\tfrac{1}{2}v^2 = P \times \frac{1}{4}\left(2\rho\frac{p}{r}\right)$$

$$= P \cdot \frac{q}{4}.$$

Also if in (2) we put $\frac{h}{p}$ for v, § 141, and $r\frac{dr}{dp}$ for ρ, we obtain

$$\frac{h^2}{p^2 r \frac{dr}{dp}} = P\frac{p}{r},$$

or

$$P = \frac{h^2}{p^3}\frac{dp}{dr},$$

which is the result contained in Art. 139.

Every apsidal line divides the orbit into two parts which are equal and similar.

For the acceleration at any point being a function of the distance from the centre of attraction, when the particle has reached an apse it must proceed to describe on the other side of the apse a path equal, similar and symmetrical with the path it has already described, and hence an apsidal line divides the orbit into two parts which are equal and similar. (Compare, however, Ex. 80 at end of Chapter.)

146. *In a central orbit there cannot be more than two apsidal distances.*

For, since the parts of the orbit on opposite sides of an apse are similar, the particle after passing two apses must come next to one at an equal distance with that of the first, then to one at an equal distance with that of the second, and so on. Hence there can be but two apsidal distances.

147. When the central attraction varies as a power of the distance, we may obtain the above result, as well as the equation for determining the apsidal distances, directly from equation (4) of § 135. Suppose $P = \mu u^n$, then we have

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2}u^{n-2} = 0.$$

Multiplying by $h^2 \dfrac{du}{d\theta}$ and integrating, we have

$$\tfrac{1}{2}h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \tfrac{1}{2}v^2 = \frac{\mu}{n-1}u^{n-1} + C.$$

Suppose the particle projected with a velocity equal to q times the velocity from infinity at the same distance, and let c be the initial value of u, then when $u = c$,

$$\tfrac{1}{2}v^2 = \frac{\mu q^2}{n-1}c^{n-1} \text{ (§ 102)};$$

whence $$C = (q^2 - 1)\frac{\mu}{n-1}c^{n-1};$$

and therefore $\frac{1}{2}h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \frac{\mu}{n-1}\{u^{n-1} + (q^2-1)c^{n-1}\}$.

To determine the apsidal distances we must put $\frac{du}{d\theta} = 0$, which gives

$$u^{n-1} - \frac{h^2(n-1)}{2\mu}u^2 + (q^2-1)c^{n-1} = 0.$$

The form of this equation shews that it can have at most two positive roots, which are therefore the two apsidal distances.

Although there can be but two apsidal distances, there may be any number of apses, and the angle between two consecutive apsidal distances is called the apsidal angle. Generally, to determine this angle, the equation of the orbit must first be found for the particular case considered; but the apsidal angle may be determined approximately for any law of attraction, without first finding the form of the orbit, if we assume that it does not differ much from a circle.

148. *A particle revolves in an orbit which is very nearly circular, and is acted on by an attraction varying as any function of the distance and directed towards the centre of the circle: to determine the apsidal angle.*

If we put P in the form $\mu u^2\phi(u)$ the differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2}\phi(u) = 0.$$

If the orbit were circular, we should have

$$u = c,$$

and $$\frac{d^2u}{d\theta^2} = 0,$$

in which case

$$c - \frac{\mu}{h^2}\phi(c) = 0 \text{ ....}(a).$$

When the orbit is very nearly circular we may put $u = c + x$, where x is always very small. Hence

$$\frac{d^2x}{d\theta^2} + c + x - \frac{\mu}{h^2}\phi(c + x) = 0,$$

or $$\frac{d^2x}{d\theta^2} + c + x - \frac{\mu}{h^2}\{\phi(c) + x\phi'(c)\} = 0, \text{ nearly};$$

and (a) enables us to reduce this to

$$\frac{d^2x}{d\theta^2} + x\left(1 - \frac{\mu\phi'(c)}{h^2}\right) = 0,$$

or, by a second application of (a),

$$\frac{d^2x}{d\theta^2} + \left\{1 - \frac{c\phi'(c)}{\phi(c)}\right\}x = 0, \qquad\qquad (b)$$

the integral of which is (§ 88)

$$x = A\cos\left[\sqrt{\left\{1 - \frac{c\phi'(c)}{\phi(c)}\right\}}\,\theta + B\right].$$

Hence the general value of θ which renders $\frac{dx}{d\theta} = 0$, is given by the equation

$$\sqrt{\left\{1 - \frac{c\phi'(c)}{\phi(c)}\right\}}\,\theta + B = n\pi,$$

n being any integer; and consequently the difference between any two such successive values of θ is

$$\frac{\pi}{\sqrt{\left\{1 - \frac{c\phi'(c)}{\phi(c)}\right\}}},$$

the approximate apsidal angle.

Thus if the attraction vary directly as the n^{th} power of the distance, we have

$$\mu u^2 \phi(u) = \mu u^{-n}; \text{ and } \phi(u) = u^{-n-2},$$

whence $$\phi'(u) = -(n+2)\, u^{-n-3},$$

and the apsidal angle is

$$\frac{\pi}{\sqrt{(3+n)}}.$$

This shews that n cannot be less than -3, or that the attraction must vary according to a lower inverse power of the distance than the third, if the circle with the centre of attraction at its centre is to be an approximation to the path of the particle: and the investigation furnishes a simple example of the determination of the conditions of *Kinetic Stability*, which we cannot discuss in this elementary treatise.

To find the law of attraction that the apsidal angle in the nearly circular orbit, *whatever be its radius*, may be equal to a given angle, α suppose, we have

$$\frac{\pi}{\sqrt{\left\{1 - \dfrac{c\phi'(c)}{\phi(c)}\right\}}} = \alpha;$$

from which $$\frac{\phi'(c)}{\phi(c)} = \frac{1}{c}\left(1 - \frac{\pi^2}{\alpha^2}\right);$$

or, by integration,

$$\log \frac{\phi(c)}{C} = \left(1 - \frac{\pi^2}{\alpha^2}\right) \log c,$$

whence $$\phi(c) = Cc^{1-\frac{\pi^2}{\alpha^2}};$$

and therefore the law of attraction, $\mu u^2 \phi(u)$, is $\mu u^{3-\frac{\pi^2}{\alpha^2}}$.

Thus for $\alpha = \pi$ we have the law of the inverse square of the distance, for $\alpha = \dfrac{\pi}{2}$ the law of the direct distance, while $\alpha = \dfrac{\pi}{\sqrt{3}}$ corresponds to a *constant* central attraction.

If $1 - \dfrac{c\phi'c}{\phi c}$ be zero or negative, the form of the integral of (*b*) above shows that x does not remain infinitely small; i.e.

that a circle is not a kinetically stable path under the conditions. In this case all that (*b*) can furnish is an account of the way in which the orbit *begins* to differ from a circle in consequence of a slight disturbance.

149. *A particle is projected from a given point in a given direction and with a given velocity, and moves under the action of a central attraction varying inversely as the square of the distance; to determine the orbit.*

We have $P = \mu u^2$, and therefore

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} = 0,$$

or

$$\frac{d^2}{d\theta^2}\left(u - \frac{\mu}{h^2}\right) + \left(u - \frac{\mu}{h^2}\right) = 0\,;$$

the integral of which is

$$u - \frac{\mu}{h^2} = A\cos(\theta + B),$$

or, as it is usually written,

$$u = \frac{\mu}{h^2}\{1 + e\cos(\theta - \alpha)\} \text{.............(1).}$$

This is the polar equation of a conic section, the focus (the centre of force) being the pole.

It gives by differentiation

$$\frac{du}{d\theta} = -\frac{\mu}{h^2}e\sin(\theta - \alpha) \text{...............(2).}$$

Let R be the distance of the point of projection from the centre; β the angle, and V the velocity, of projection; then when $\theta = 0$,

$$u = \frac{1}{R}, \qquad \cot\beta = -\left(\frac{1}{u}\frac{du}{d\theta}\right)_{\theta=0}.$$

Hence, by (1), $$\frac{h^2}{\mu R} - 1 = e\cos\alpha,$$

and by (2), $$\frac{h^2}{\mu R}\cot\beta = -e\sin\alpha.$$

From these, $\tan\alpha = \dfrac{h^2 \cot\beta}{\mu R - h^2}$ ....(3),

and $$e^2 = \frac{h^4}{\mu^2 R^2}\operatorname{cosec}^2\beta - \frac{2h^2}{\mu R} + 1 \text{ ....(4).}$$

But $h^2 = V^2R^2\sin^2\beta$, § 142;

wherefore $$\tan\alpha = \frac{V^2R\sin\beta\cos\beta}{\mu - V^2R\sin^2\beta} \text{ ....(3')}$$

and $$1 - e^2 = \frac{V^2R^2\sin^2\beta}{\mu}\left(\frac{2}{R} - \frac{V^2}{\mu}\right) \text{ ....(4').}$$

Now (1) is the general polar equation of a conic section focus the pole; and, as its nature depends on the value of the excentricity e given by (4′), we see that

if $V^2 > \dfrac{2\mu}{R}$, $e > 1$, and the orbit is a hyperbola,

$V^2 = \dfrac{2\mu}{R}$, $e = 1$, ... a parabola,

$V^2 < \dfrac{2\mu}{R}$, $e < 1$, ... an ellipse.

150. By § 102, the square of the velocity from infinity at distance R, for the law of attraction we are considering, is $\dfrac{2\mu}{R}$, and the above conditions may therefore be expressed more concisely by saying that the orbit will be a hyperbola, a parabola, or an ellipse, according as the velocity of projection is greater than, equal to, or less than, the velocity from infinity. Illustrations of this proposition are found in the cases of comets and meteor swarms.

The velocity of a particle moving in a circle is also often taken as the standard of comparison for estimating the velocities of bodies in their orbits. For the gravitation law of attraction the square of the velocity in a circle of radius R is $\dfrac{\mu}{R}$; and the above conditions may be expressed in another

form by saying that the orbit will be a hyperbola, a parabola, or an ellipse, according as the velocity of projection is greater than, equal to, or less than, $\sqrt{2}$ times the velocity in a circle at the same distance.

151. Supposing the orbit to be an ellipse, we shall obtain its major axis and latus rectum most easily by a different process of integrating the differential equation. Multiplying it by $h^2 \frac{du}{d\theta}$ and integrating, we obtain

$$\frac{1}{2} h^2 \left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \frac{1}{2} v^2 = C + \mu u.$$

But when $u = \frac{1}{R}$, $v = V$; which gives

$$C = \frac{1}{2} V^2 - \frac{\mu}{R};$$

hence $$\frac{1}{2} h^2 \left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \frac{1}{2} v^2 = \frac{1}{2} V^2 - \frac{\mu}{R} + \mu u \quad \text{.....(5)}.$$

Now to determine the apsidal distances, we must put

$$\frac{du}{d\theta} = 0;$$

and this gives us the condition

$$u^2 - \frac{2\mu}{h^2} u + \frac{2\mu}{h^2 R} - \frac{V^2}{h^2} = 0 \quad \text{.............(6)},$$

which is a quadratic equation whose roots are the reciprocals of the two apsidal distances. But if a be the semi-axis major, and e the excentricity, these distances are

$$a(1-e) \text{ and } a(1+e).$$

Hence, as the coefficient of the second term of (6) is the sum of the roots with their signs changed, we have

$$\frac{1}{a(1-e)} + \frac{1}{a(1+e)} = \frac{2\mu}{h^2};$$

or $$a(1-e^2) = \frac{h^2}{\mu} \quad \text{.....................(7)}.$$

And, as the third term is the product of the roots,

$$\frac{1}{a^2(1-e^2)} = \frac{2\mu}{h^2R} - \frac{V^2}{h^2}:$$

or

$$\frac{1}{a} = \frac{2}{R} - \frac{V^2}{\mu} \quad \text{.....................(8)},$$

or

$$\frac{1}{2}V^2 = \frac{\mu}{R} - \frac{\mu}{2a},$$

and therefore

p " 1

$$\frac{1}{2}v^2 = \frac{\mu}{r} - \frac{\mu}{2a} \quad \text{.....................(9)}.$$

Equations (7) and (8) give the latus rectum and major axis of the orbit, and shew that the major axis is independent of the direction of projection.

Equation (9) gives a useful expression for the velocity at any point, and shews that the radius of the circle of zero velocity is $2a$.

152. The time of describing any given angle is to be obtained from the formula,

$$\begin{aligned} r^2\frac{d\theta}{dt} &= h \\ &= \sqrt{\{\mu a(1-e^2)\}}, \qquad \text{by equation (7).} \end{aligned}$$

From this, combined with the polar equation of a conic section about the focus, we have

$$\begin{aligned} \frac{dt}{d\theta} &= \frac{r^2}{\sqrt{\{\mu a(1-e^2)\}}} \\ &= \sqrt{\left(\frac{a^3(1-e^2)^3}{\mu}\right)}\frac{1}{(1+e\cos\theta)^2}; \end{aligned}$$

measuring the angle from the nearest apse. To integrate this, let

$$\Theta = \frac{\sin\theta}{1+e\cos\theta}.$$

Then
$$\frac{d\Theta}{d\theta}=\frac{\cos\theta+e}{(1+e\cos\theta)^2}=\frac{\frac{1}{e}(1+e\cos\theta)-\frac{1-e^2}{e}}{(1+e\cos\theta)^2}$$
$$=\frac{1}{e}\frac{1}{1+e\cos\theta}-\frac{1-e^2}{e}\frac{1}{(1+e\cos\theta)^2};$$
$$\therefore\int\frac{d\theta}{(1+e\cos\theta)^2}=-\frac{e\Theta}{1-e^2}+\frac{1}{1-e^2}\int\frac{d\theta}{1+e\cos\theta}$$ 305

$$=-\frac{e}{1-e^2}\frac{\sin\theta}{1+e\cos\theta}+\frac{1}{1-e^2}\int\frac{\sec^2\frac{\theta}{2}\,d\theta}{(1+e)+(1-e)\tan^2\frac{\theta}{2}};$$
$$=-\frac{e}{1-e^2}\frac{\sin\theta}{1+e\cos\theta}+\frac{2}{(1-e^2)^{\frac{3}{2}}}\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\},$$ 49

(if $e<1$);

$$\text{or }=\frac{e}{e^2-1}\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{(e^2-1)^{\frac{3}{2}}}\log\left\{\frac{\sqrt{(e+1)}\cos\frac{\theta}{2}+\sqrt{(e-1)}\sin\frac{\theta}{2}}{\sqrt{(e+1)}\cos\frac{\theta}{2}-\sqrt{(e-1)}\sin\frac{\theta}{2}}\right\}$$

(if $e>1$).

Hence the time of describing, about the focus, an angle θ measured from the nearer apse is, in the ellipse,

$$\sqrt{\frac{a^3}{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}-e\sqrt{(1-e^2)}\frac{\sin\theta}{1+e\cos\theta}\right];$$

that is, $\frac{2}{h}$ of the sectorial area ASP (figure to § 160); and, in the hyperbola,

$$\sqrt{\frac{a^3}{\mu}}\left[\log\left\{\frac{\sqrt{(e+1)}\cos\frac{\theta}{2}-\sqrt{(e-1)}\sin\frac{\theta}{2}}{\sqrt{(e+1)}\cos\frac{\theta}{2}+\sqrt{(e-1)}\sin\frac{\theta}{2}}\right\}+e\sqrt{(e^2-1)}\frac{\sin\theta}{1+e\cos\theta}\right],$$

that is, $\frac{2}{h}$ of the sectorial area ASP of the hyperbola.

Hence these expressions for the time through any area of an elliptic or hyperbolic orbit about a focus might have been written down from the known expressions for the area of an elliptic or hyperbolic sector.

153. In the parabola, if d be the apsidal distance, the integral becomes

$$\{\text{since } e=1,\ a(1-e)=d,\ a(1-e^2)=2d\},$$

$$t=\sqrt{\frac{8d^3}{\mu}}\int\frac{d\theta}{(1+\cos\theta)^2}$$

$$=\sqrt{\frac{8d^3}{\mu}}\int\frac{1}{4}\sec^4\frac{\theta}{2}\,d\theta$$

$$=\sqrt{\frac{2d^3}{\mu}}\int\left(1+\tan^2\frac{\theta}{2}\right)d\tan\frac{\theta}{2}$$

$$=\sqrt{\frac{2d^3}{\mu}}\left(\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right).$$

154. From the result for the ellipse we see that the periodic time is $2\pi\sqrt{\dfrac{a^3}{\mu}}$. This might also have been found from the consideration of equable description of areas by the radius vector.

Thus $$T=\frac{2\text{ area of ellipse}}{h}$$

$$=\frac{2\pi a^2\sqrt{(1-e^2)}}{\sqrt{\{\mu a(1-e^2)\}}}$$

$$=2\pi\sqrt{\frac{a^3}{\mu}}.$$

In the notation commonly employed we write

$$T=\frac{2\pi}{n},$$

where n, which is called the *Mean Motion*, is

$$\sqrt{\frac{\mu}{a^3}}.$$

155. By laborious calculation from an immense series of observations of the planets, and of Mars in particular, Kepler was led to enuntiate the following as the laws of the planetary motions about the Sun.

I. The planets describe, relatively to the Sun, Ellipses of which the Sun occupies a focus.

II. The radius vector of each planet traces out equal areas in equal times.

47 III. The squares of the periodic times of any two planets are as the cubes of the major axes of their orbits.

156. From the second of these laws we conclude that the planets are retained in their orbits by an attraction tending to the Sun. For,

If the radius vector of a particle moving in a plane describe equal areas in equal times about a point in that plane, the resultant attraction on the particle tends to that point.

Take the point as origin, and let x, y be the co-ordinates of the particle at time t; X, Y the component accelerations due to the attraction acting on it, resolved parallel to the axes; the equations of motion are

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y \text{.....................} (1).$$

But by hypothesis, if A be the area traced out by the radius vector, $\frac{dA}{dt}$ is constant.

Hence, $$2\frac{dA}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt} = C.$$

Differentiating, $$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

or, by (1), $$xY - yX = 0.$$

Hence, $$\frac{Y}{X} = \frac{y}{x},$$

and by the parallelogram of forces (§ 67) the resultant of X and Y passes through the origin.

157. From the first of these laws it follows that the law of the intensity of the attraction is that of the inverse square of the distance.

The polar equation of an Ellipse referred to its focus is

$$u = \frac{2}{l}(1 + e\cos\theta),$$

where l is the latus rectum.

Hence, $$\frac{d^2u}{d\theta^2} = -\frac{2e}{l}\cos\theta,$$

and therefore the attraction to the focus requisite for the description of the ellipse is (§ 135)

$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$
$$= \frac{2h^2}{l}u^2.$$

Hence, *if the orbit be an ellipse, described about a centre of attraction at the focus, the law of intensity is that of the inverse square of the distance.*

158. From the third it follows that the attraction of the Sun (supposed fixed) which acts on unit of mass of each of the planets is the same for each planet at the same distance.

For, in the formula in § 154, T^2 will not vary as a^3 unless μ be constant, i.e. unless the strength of attraction of the Sun be the same for all the planets.

We shall find afterwards that for more reasons than one Kepler's laws are only approximate, but their enuntiation was sufficient to enable Newton to propound the doctrine of Universal Gravitation; viz. that *every particle of matter in*

the universe attracts every other with an attraction whose direction is that of the line joining them and whose magnitude is as the product of the masses directly, and as the square of the distance inversely; or according to Maxwell's "Matter and Motion," *between every pair of particles there is a stress of the nature of a tension, proportional to the product of the masses of the particles divided by the square of their distance.*

On this hypothesis, neglecting the mutual attractions of the planets, Kepler's third law should be stated (Chap. XI.): *The cubes of the major axes of the orbits are as the squares of the periodic times and the sums of the masses of the Sun and the planet.*

159. Suppose APA' to be an elliptic orbit described about a centre of attraction in the focus S. Also suppose P to be the position of the particle at any time t. Draw PM perpendicular to the major axis ACA', and produce it to cut the auxiliary circle in the point Q. Let C be the common centre of the curves. Join CQ.

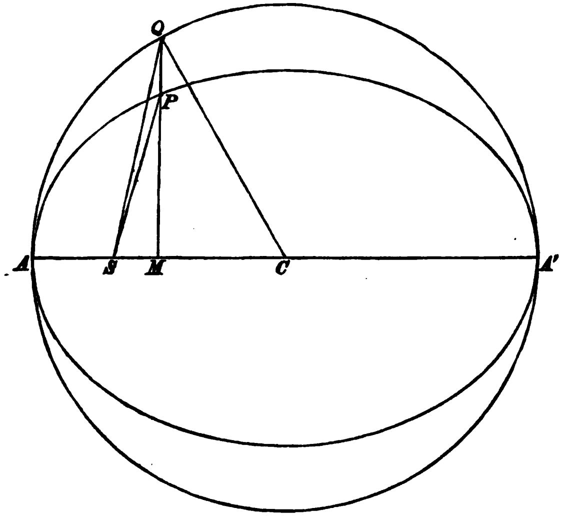
When the moving particle is at A, the nearest point of the orbit to S, it is said to be in *Perihelion.*

The angle ASP, or the excess of the particle's longitude over that of the perihelion, is called the *True Anomaly.* Let us denote it by θ.

The angle ACQ is called the *Excentric Anomaly,* and is generally denoted by u. And if $\frac{2\pi}{n}$ be the time of a complete revolution, nt is the circular measure of an imaginary angle called the *Mean Anomaly;* it would evidently be the true anomaly if the particle's angular velocity about S were constant.

160. It is easy from known properties of the ellipse to deduce relations between the mean and excentric, and also between the true and excentric, anomalies; this we proceed to do.

To find the relation between the mean and excentric anomalies.



In the figure QCA is the excentric anomaly, and the mean anomaly is evidently to 2π as the area PSA is to the whole area of the elliptic orbit (§§ 154—159), or as area QSA to area of auxiliary circle.

Now area QSA = area QCA − area QCS

$$= \tfrac{1}{2}a^2u - \tfrac{1}{2}a \, . \, ae \, . \sin u$$

(a being the major semi-axis of the orbit and e the excentricity)

$$= \frac{a^2}{2}(u - e\sin u).$$

Hence $$\frac{nt}{2\pi} = \frac{\frac{a^2}{2}(u-e\sin u)}{\pi a^2};$$

or $$nt = u - e\sin u.$$

161. *To find the relation between the true and excentric anomalies.*

We have (by Conics)

$$SP = \frac{a(1-e^2)}{1+e\cos\theta}.$$

But $$SP = a - eCM = a(1 - e\cos u).$$

Hence $$\frac{1-e^2}{1+e\cos\theta} = 1 - e\cos u.$$

Hence $$\cos\theta = \frac{\cos u - e}{1 - e\cos u},$$

and
$$\begin{aligned}
\tan\frac{\theta}{2} &= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \sqrt{\frac{1 - e\cos u - \cos u + e}{1 - e\cos u + \cos u - e}} \\
&= \sqrt{\frac{(1+e)(1-\cos u)}{(1-e)(1+\cos u)}} \\
&= \sqrt{\left(\frac{1+e}{1-e}\right)}\tan\frac{u}{2};
\end{aligned}$$

therefore $$\tan\frac{u}{2} = \sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2},$$

$$\sin u = \sqrt{1-e^2}\,\frac{\sin\theta}{1+e\cos\theta},$$

substituted in $nt = u - e\sin u$ give the expressions obtained in § 152.

162. By far the most important problem is to find the values of θ and r as functions of t, so that the direction and length of a planet's radius vector may be determined for any given time. This generally goes by the name of Kepler's Problem.

Before entering on the systematic development of u, r and θ in terms of t from our equations, it may be useful to

remark, that if e be so small that higher terms than its square may be neglected, we may easily obtain developments correct to the first three terms.

Thus
$$u = nt + e \sin u$$
$$= nt + e \sin (nt + e \sin nt) \text{ nearly},$$
$$= nt + e \sin nt + \frac{e^2}{2} \sin 2nt.$$

Also
$$\frac{r}{a} = 1 - e \cos u$$
$$= 1 - e \cos (nt + e \sin nt)$$
$$= 1 - e \cos nt + \frac{e^2}{2} (1 - \cos 2nt).$$

And
$$r^2 \frac{d\theta}{dt} = \sqrt{\{\mu a (1 - e^2)\}},$$
which may be written (§ 154)
$$\frac{a^2 (1 - e^2)^2}{(1 + e \cos \theta)^2} \frac{d\theta}{dt} = na^2 (1 - e^2)^{\frac{1}{2}},$$
or
$$(1 - e^2)^{\frac{3}{2}} (1 + e \cos \theta)^{-2} \frac{d\theta}{dt} = n.$$

Keeping powers of e lower than the third
$$\left(1 - 2e \cos \theta + \frac{3}{2} e^2 \cos 2\theta\right) \frac{d\theta}{dt} = n,$$
or
$$nt = \theta - 2e \sin \theta + \frac{3}{4} e^2 \sin 2\theta;$$
whence $\theta = nt + 2e \sin \theta - \dfrac{3}{4} e^2 \sin 2\theta$
$$= nt + 2e \sin (nt + 2e \sin nt) - \frac{3}{4} e^2 \sin 2nt$$
$$= nt + 2e \sin nt + 4e^2 \cos nt \sin nt - \frac{3}{4} e^2 \sin 2nt$$
$$= nt + 2e \sin nt + \frac{5}{4} e^2 \sin 2nt.$$

163. Kepler's Problem. *To find* r *and* θ *as functions of* t *from the equations*

$$r = a(1 - e\cos u)\ldots\ldots\ldots\ldots(1);$$

$$\tan\frac{\theta}{2} = \sqrt{\left(\frac{1+e}{1-e}\right)}\tan\frac{u}{2}\ldots\ldots\ldots\ldots(2);$$

$$nt = u - e\sin u\ \ldots\ldots\ldots\ldots(3).$$

These equations evidently give r, θ, and t directly for any assigned value of u, but this is of little value in practice. The method of solution which we proceed to give is that of Lagrange, and the general principle of it is this—

We can develop θ from equation (2) in a series ascending by powers of a small quantity, a function of e, the coefficients of these powers involving u and the sines of multiples of u. Now by Lagrange's theorem we may from equation (3) express u, $1 - e\cos u$, $\sin u$, $\sin 2u$, &c. in series ascending by powers of e, whose coefficients are sines or cosines of multiples of nt. Hence by substituting these values in equation (1) and in the development of (2), we have r and θ expressed in series whose terms rapidly decrease, and whose coefficients are sines or cosines of multiples of nt. This is the complete practical solution of the problem.

164. *To express the true, as a function of the excentric, anomaly.*

Substituting in (2) the exponential expressions for the tangents, and writing i for $\sqrt{-1}$, we have

$$\frac{\epsilon^{\frac{i\theta}{2}} - \epsilon^{-\frac{i\theta}{2}}}{\epsilon^{\frac{i\theta}{2}} + \epsilon^{-\frac{i\theta}{2}}} = \sqrt{\left(\frac{1+e}{1-e}\right)}\frac{\epsilon^{\frac{iu}{2}} - \epsilon^{-\frac{iu}{2}}}{\epsilon^{\frac{iu}{2}} + \epsilon^{-\frac{iu}{2}}},$$

whence

$$\epsilon^{i\theta} = \frac{\epsilon^{iu}\{\sqrt{(1+e)} + \sqrt{(1-e)}\} + \{\sqrt{(1-e)} - \sqrt{(1+e)}\}}{\epsilon^{iu}\{\sqrt{(1-e)} - \sqrt{(1+e)}\} + \{\sqrt{(1-e)} + \sqrt{(1+e)}\}};$$

$$\text{or, putting } \lambda = \frac{\sqrt{(1+e)} - \sqrt{(1-e)}}{\sqrt{(1+e)} + \sqrt{(1-e)}} = \frac{e}{1 + \sqrt{(1-e^2)}},$$

$$\epsilon^{i\theta} = \epsilon^{iu}\cdot\frac{1 - \lambda\epsilon^{-iu}}{1 - \lambda\epsilon^{iu}}.$$

Taking the logarithm of each side and dividing by i,

$$\theta = u + \frac{\lambda}{i}\{\epsilon^{iu} - \epsilon^{-iu}\} + \frac{\lambda^2}{2i}\{\epsilon^{2iu} - \epsilon^{-2iu}\} + \dots$$

$$= u + 2\left(\lambda \sin u + \frac{\lambda^2}{2}\sin 2u + \frac{\lambda^3}{3}\sin 3u + \&c.\right)\dots\dots\dots(4).$$

165. *To develop* u *in terms of* t.

If we have

$$y = z + x\phi(y)\dots\dots\dots\dots\dots\dots\dots(5),$$

we obtain, by Lagrange's Theorem, the development

$$f(y) = f(z) + x\phi(z) f'(z) + \frac{x^2}{1.2}\frac{d}{dz}\{\overline{\phi(z)}|^2 f'(z)\}$$

$$+ \frac{x^3}{1.2.3}\left(\frac{d}{dz}\right)^2\{\overline{\phi(z)}|^3 f'(z)\} + \&c.\dots\dots\dots\dots\dots(6).$$

Now equation (3) may be put in the form

$$u = nt + e\sin u,$$

which is identical with (5) if

$$y = u, \quad z = nt, \quad x = e, \text{ and } \phi(y) = \sin y.$$

Also, as it is the development of u that we require, we must put

$$f(u) = u, \text{ and } f'(u) = 1. \quad \text{Hence, by (6)}$$

$$y = z + x\sin z + \frac{x^2}{1.2}\frac{d}{dz}(\sin^2 z) + \frac{x^3}{1.2.3}\left(\frac{d}{dz}\right)^2(\sin^3 z) + \&c.;$$

and, substituting for the powers of $\sin z$ their corresponding expressions in sines and cosines of multiples of z,

$$y = z + x\sin z + \frac{x^2}{1.2}\frac{d}{dz}\left(\frac{1-\cos 2z}{2}\right) + \frac{x^3}{1.2.3}\left(\frac{d}{dz}\right)^2\left(\frac{3\sin z - \sin 3z}{4}\right)$$

$$+ \frac{x^4}{1.2.3.4}\left(\frac{d}{dz}\right)^3\left(\frac{3 - 4\cos 2z + \cos 4z}{8}\right) + \&c.$$

$$= z + x\sin z + \frac{x^2}{2}\sin 2z + \frac{x^3}{8}(3\sin 3z - \sin z) + \dots\dots$$

or, substituting for x, y, z their values as above,

$$u = nt + e\sin nt + \frac{e^2}{2}\sin 2nt + \frac{e^3}{8}(3\sin 3nt - \sin nt)$$
$$+ \frac{e^4}{6}(2\sin 4nt - \sin 2nt) + \&c. \ldots\ldots\ldots\ldots (7).$$

To develop sin u, we recur to equation (3), which gives, after the elimination of u by means of (7),

$$\sin u = \sin nt + \frac{e}{2}\sin 2nt + \frac{e^2}{8}(3\sin 3nt - \sin nt) + \&c. \ldots (8).$$

By the application of Lagrange's theorem to equation (3), it is easy to deduce the following expressions:

$$\sin 2u = \sin 2nt + e(\sin 3nt - \sin nt) + e^2(\sin 4nt - \sin 2nt)$$
$$+ \frac{e^3}{24}(4\sin nt - 27\sin 3nt + 25\sin 5nt) + \&c.$$

$$\sin 3u = \sin 3nt + \frac{3e}{2}(\sin 4nt - \sin 2nt)$$
$$+ \frac{e^2}{8}(15\sin 5nt - 18\sin 3nt + 3\sin nt) + \&c.$$

&c. = &c.

Substituting these values in (4), we obtain the value of θ, containing however the quantity λ. If we take as its approximate value $\frac{e}{2} + \frac{e^3}{8}$, and make the requisite substitutions, we obtain

$$\theta = nt + (2e - \frac{1}{4}e^3)\sin nt + \frac{5}{4}e^2\sin 2nt + \frac{13}{12}e^3\sin 3nt + \ldots.$$

which is correct as far as e^3.

[The development of u in terms of t is

$$u = nt + 2\Sigma_{m=1}^{m=\infty}\frac{1}{m}J_m(me)\sin m\,nt,$$

$$\text{where } J_m(me) = \frac{1}{\pi}\int_0^\pi \cos m(t - e\sin t)\,dt$$

is Bessel's function of the n^{th} order.]

For the development of r and θ in terms of t, the coefficients being Bessel's functions, see Todhunter's *Treatise on Legendre's, Laplace's, and Bessel's Functions.*

166. In proceeding farther with the development, it becomes necessary to expand λ and its powers in series ascending by powers of e. This is readily done as follows.

We have

$$\lambda = \frac{e}{1+\sqrt{(1-e^2)}} = \frac{e}{E} \text{ suppose.}$$

Hence
$$E = 2 - \frac{e^2}{E},$$

from which, by Lagrange's Theorem,

$$E^{-p} = \frac{1}{2^p} + \frac{p}{2^{p+2}}e^2 + \frac{p\,.\,(p+3)}{2\,.\,2^{p+4}}e^4 + \&c.;$$

and thus the value of λ^p, being $e^p E^{-p}$, is known.

The correct value of θ to the fifth power of e is thus found to be

$$nt + 2e\sin nt + \frac{5e^2}{4}\sin 2nt + \frac{e^3}{2^2\,.\,3}(13\sin 3nt - 3\sin nt)$$

$$+\frac{e^4}{2^5\,.\,3}(103\sin 4nt - 44\sin 2nt)$$

$$+\frac{e^5}{2^6\,.\,3\,.\,5}(1097\sin 5nt - 645\sin 3nt + 50\sin nt).$$

167. *To develop* r *in terms of* t.

From (1) it is evident that all we have to do is to develop by Lagrange's Theorem, $1 - e\cos u$ as a function of t, from $nt = u - e\sin u$.

To develop $(1 - e\cos u)$ *in terms of* t.

Here $f(y) = 1 - e\cos y$,

$f'(y) = e\sin y$;

and the form of ϕ is the same as before; hence

$$1 - e\cos y = (1 - e\cos z) + x\sin z\,(e\sin z)$$
$$+ \frac{x^2}{1.2}\frac{d}{dz}(\sin^2 z \,.\, e\sin z) + \ldots\ldots\ldots$$

Hence, as before, substituting for the powers of sines their equivalent expressions in sines and cosines of multiple arcs, differentiating, and substituting u for y, nt for z, and e for x, we have

$$1 - e\cos u = \frac{r}{a} = 1 - e\cos nt + \frac{e^2}{2}(1 - \cos 2nt)$$
$$+ \frac{e^3}{8}(3\cos nt - 3\cos 3nt)$$
$$+ \frac{e^4}{3}(\cos 2nt - \cos 4nt) + \&c.$$

which gives the radius vector in terms of the time.

168. Lambert's Theorem. *The area of a focal elliptic sector and therefore the time through any arc of the ellipse, described about the focus, can be expressed in terms of the chord and the focal distances of the ends of the arc.*

If r_1, r_2 be the focal distances of the ends and c the chord of the arc, it is proved in Williamson's *Integral Calculus*, § 137, that the sectorial area is

$$\tfrac{1}{2}ab\,\{\phi_1 - \phi_2 - (\sin\phi_1 - \sin\phi_2)\},$$

where ϕ_1 and ϕ_2 are given by the equations

$$\sin\tfrac{1}{2}\phi_1 = \frac{1}{2}\sqrt{\left(\frac{r_1 + r_2 + c}{a}\right)},\ \sin\tfrac{1}{2}\phi_2 = \frac{1}{2}\sqrt{\left(\frac{r_1 + r_2 - c}{a}\right)};$$

and therefore if t denote the time in the arc,

$$nt = \phi_1 - \phi_2 - (\sin\phi_1 - \sin\phi_2).$$

EXAMPLES.

(1) A particle describes an ellipse under an attraction always directed to the centre, to determine the law of the attraction.

From the polar equation of the ellipse, centre pole,

$$u^2 = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\text{; we have } u\frac{du}{d\theta} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos\theta\sin\theta\,;$$

$$\therefore u\frac{d^2u}{d\theta^2} + \left(\frac{du}{d\theta}\right)^2 = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)(\cos^2\theta - \sin^2\theta)\,;$$

$$\therefore P = \frac{h^2}{u}\left(u^4 + u^3\frac{d^2u}{d\theta^2}\right)$$

$$= \frac{h^2}{u}\left[u^4 - u^2\left(\frac{du}{d\theta}\right)^2 + u^2\left\{u\frac{d^2u}{d\theta^2} + \left(\frac{du}{d\theta}\right)^2\right\}\right]$$

$$= \frac{h^2}{u}\left\{u^4 - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2\cos^2\theta\,\sin^2\theta + u^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\left(\cos^2\theta - \sin^2\theta\right)\right\}$$

$$= \frac{h^2}{u}\left\{u^2 + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos^2\theta\right\}\left\{u^2 - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin^2\theta\right\}$$

$$= \frac{h^2}{u}\cdot\frac{1}{b^2}\cdot\frac{1}{a^2} = \frac{h^2}{a^2b^2}r\,;$$

and therefore the law is that of the direct distance.

(2) A particle describes a conic section under an attraction always directed to one of the foci, to find the law of attraction.

In this case

$$u = \frac{1}{a(1-e^2)}\{1 + e\cos(\theta - \alpha)\},$$

and therefore
$$P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= \frac{h^2u^2}{a(1-e^2)} \propto \frac{1}{r^2}.$$

(3) Find the attraction to the pole under which a particle may describe an equiangular spiral.

$$P \propto \frac{1}{r^3}.$$

(4) Find the attraction by which a particle may describe the lemniscate of Bernouilli, the centre being the node.

$$P \propto \frac{1}{r^7}.$$

(5) Find the attraction by which a particle may describe a circle, the centre of attraction being in the circumference of the circle.

$$P \propto \frac{1}{r^5}.$$

(6) Find the attraction to the pole under which a particle will describe the curve

$$r^n = a^n \cos n\theta,$$

and interpret the result when $n = -1$. Deduce the law of attraction for (1) a rectangular hyperbola, (2) a lemniscate, (3) a circle about a point in a circumference, (4) a cardioid, (5) a parabola.

(7) Prove that the attraction to the pole under which a particle will describe the n^{th} pedal of a cardioid varies as $r^{-\frac{3n+8}{n+2}}$. Deduce the law of attraction for a circle about a point on the circumference.

(8) A particle is projected from a given point in a given direction with the velocity from an infinite distance, and is under an attraction varying inversely as the n^{th} power of the distance, to determine the orbit.

Here $P = \mu u^n$, and therefore

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} u^{n-2} = 0.$$

Multiplying by $h^2 \frac{du}{d\theta}$ and integrating,

$$\frac{1}{2} h^2 \left\{ \left(\frac{d\theta}{du}\right)^2 + u^2 \right\} = \frac{1}{2} v^2 = \mu \int_0^u u^{n-2}\, du = \frac{\mu u^{n-1}}{n-1},$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2\mu u^{n-1}}{h^2 (n-1)}.$$

Now if a be the apsidal distance,

$$h^2 = v^2 a^2 = \frac{2\mu a^{1-n}}{n-1} a^2 = \frac{2\mu a^{3-n}}{n-1};$$

therefore $$\left(\frac{du}{d\theta}\right)^2 + u^2 = a^{n-3} u^{n-1},$$

$$\left(\frac{du}{d\theta}\right)^2 = u^2 \{(au)^{n-3} - 1\},$$

$$\frac{d\theta}{du} = \frac{1}{u\sqrt{(au)^{n-3} - 1}};$$

integrating $$\frac{n-3}{2} \theta = \sec^{-1} (au)^{\frac{n-3}{2}},$$

or $$r^{\frac{n-3}{2}} = a^{\frac{n-3}{2}} \cos \frac{n-3}{2} \theta,$$

the polar equation of the required orbit.

(9) A particle, under an attraction varying inversely as the cube of the distance, is projected from a given point with any velocity in any direction; to classify the paths described according to the circumstances of projection. The curves in question are called *Cotes' Spirals*.

The equation of motion is

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} u = 0 \text{.......................(1).}$$

The integral of this equation involves exponential or circular functions according as $\frac{\mu}{h^2}$ is greater or less than

unity, that is, according as the velocity at an apse is less or greater than the velocity from infinity.

I. Let $\frac{\mu}{h^2}$ be > 1, and let $\frac{\mu}{h^2} - 1 = k^2$; then

$$\frac{d^2u}{d\theta^2} - k^2 u = 0,$$

the integral of which is

$$u = A\epsilon^{k\theta} + B\epsilon^{-k\theta} \quad\text{..................(2)}.$$

SPECIES 1. Let A and B have the same sign; then

$$u = A\epsilon^{k\theta} + B\epsilon^{-k\theta};$$

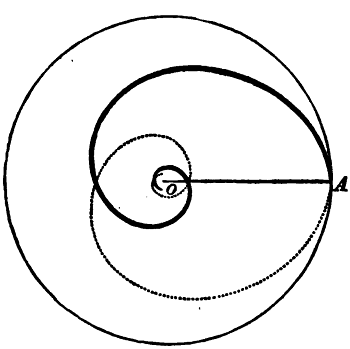
and
$$\frac{du}{d\theta} = k\,(A\epsilon^{k\theta} - B\epsilon^{-k\theta}).$$

The values of A and B may in these equations be expressed in terms of the initial distance, and angle of projection; but we may put the equation of the curve in a simpler form as follows. Let α be the value of θ corresponding to an apse, then when $\theta = \alpha$, $\frac{du}{d\theta} = 0$;

or
$$0 = A\epsilon^{k\alpha} - B\epsilon^{-k\alpha},$$

which always gives a possible value of α; and therefore

$$A\epsilon^{k\alpha} = B\epsilon^{-k\alpha} = \frac{1}{2a}, \text{ suppose.}$$



Substituting, $au = \frac{1}{2}\{\epsilon^{k(\theta-\alpha)} + \epsilon^{-k(\theta-\alpha)}\}$.

Hence when $\theta = \alpha$, $au = 1$, or a is the apsidal distance. As θ increases, u increases, or r diminishes; and when $\theta = \infty$, $u = \infty$, or $r = 0$. Hence the curve forms an infinite number of convolutions about the pole; and, as it is symmetrical on both sides of the apse, it must be as represented in the figure, where A is the apse and O the centre of attraction.

SPECIES 2. Let $\frac{\mu}{h^2} > 1$, $B = 0$, then the equation (2) becomes

$$au = \epsilon^{k\theta},$$

the equation of the logarithmic spiral. The nature of the curve will be the same if A, instead of B, vanish.

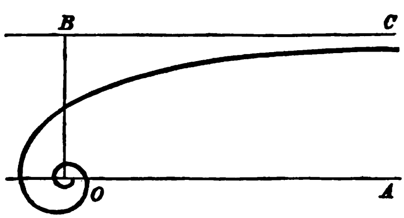
SPECIES 3. Let $\frac{\mu}{h^2} > 1$, and B negative, then by equation (2),

$$u = A\epsilon^{k\theta} - B\epsilon^{-k\theta}.$$

Putting $u = 0$, when $\theta = \alpha$, we obtain as for Species 1,

$$au = \frac{1}{2}\{\epsilon^{k(\theta-\alpha)} - \epsilon^{-k(\theta-\alpha)}\}.$$

Hence, when $\theta = \alpha$, $u = 0$ or $r = \infty$. As θ increases r decreases, and when θ is infinite $r = 0$; so that there is an



infinite number of convolutions round the pole. The curve has an asymptote parallel to OA, at a distance $\frac{a}{k}$.

II. SPECIES 4. Let $\frac{\mu}{h^2}=1$, then equation (1) becomes

$$\frac{d^2u}{d\theta^2}=0,$$

the integral of which is

$$au=\theta-\alpha,$$

the equation of the reciprocal spiral.

III. SPECIES 5. Let $\frac{\mu}{h^2}<1$, and let $1-\frac{\mu}{h^2}=k^2$, then by equation (1),

$$\frac{d^2u}{d\theta^2}+k^2u=0,$$

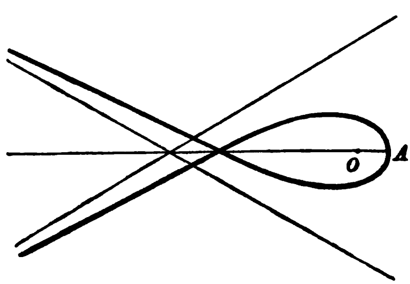
the integral of which is

$$au=\cos k(\theta-\alpha);$$

whence

$$a\frac{du}{d\theta}=-k\sin k(\theta-\alpha).$$

Then α is the value of θ corresponding to an apse, and a is the apsidal distance. The asymptotes to this curve are easily found for any assigned value of k. One case is represented in the annexed figure.



(10) A particle of mass m under a central repulsion $\frac{m\mu}{r^3}$ is projected from an apse at a distance a with velocity $\frac{\sqrt{\mu}}{2a\sqrt{2}}$. Find the orbit, and prove that the time from the apse to the distance $a\sqrt{2}$ is $\frac{2}{3}\sqrt{2}\,\frac{a^2}{\sqrt{\mu}}$.

(11) A particle under an attraction inversely proportional to the fourth power of the distance from a centre is projected in any manner; for instance, from an apse with velocity n times the velocity from infinity: determine the orbit.

(12) A particle under a central attraction varying inversely as the fifth power of the distance is projected in any manner, determine the orbit.

Here $P = \mu u^5$, and we have

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2}u^3 = 0;$$

whence $$\frac{1}{2}h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \frac{v^2}{2} = \frac{1}{4}\mu u^4 + C.$$

If the particle be projected from an apse at a distance a with velocity n times the velocity from infinity, then

$$\frac{1}{2}v^2 = n^2\mu\int_0^{\frac{1}{a}} u^3 du = \frac{1}{4}\frac{n^2\mu}{a^4};$$

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and therefore $$C = \frac{1}{4}(n^2 - 1)\frac{\mu}{a^4};$$

and $$h^2 = v^2a^2 = \frac{1}{2}\frac{n^2\mu}{a^2}.$$

Therefore $$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{a^2u^4}{n^2} + \frac{n^2-1}{n^2a^2},$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{a^2}{n^2}\left(u^4 - \frac{n^2u^2}{a^2} + \frac{n^2-1}{a^4}\right)$$

$$= \frac{a^2}{n^2}\left(u^2 - \frac{1}{a^2}\right)\left(u^2 - \frac{n^2-1}{a^2}\right),$$

or $$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^2}{n^2}\left(1 - \frac{r^2}{a^2}\right)\left\{1 - (n^2-1)\frac{r^2}{a^2}\right\};$$

and therefore r is an elliptic function of θ.

For instance, suppose $n < 1$, we have

$$r = a \operatorname{cn} m\theta,$$

where $$m^2 = \frac{2-n^2}{n^2} \text{ and } k^2 = \frac{1-n^2}{2-n^2}.$$

(13) A body moves under a central attraction

$$\frac{\mu}{c^2}\{(a^2+b^2+c^2)u^3 - 2a^2b^2u^5\},$$

being projected from an apse whose distance is a ($> b$) with a velocity $\frac{\sqrt{\mu}}{a}$, shew that it will proceed to describe the orbit whose equation is

$$r^2 = a^2 \operatorname{cn}^2 \frac{a\theta}{c} + b^2 \operatorname{sn}^2 \frac{a\theta}{c},$$

the modulus of the elliptic functions being the excentricity of an ellipse whose semi-axes are a and b.

This may be written $r = a \operatorname{dn} \frac{a\theta}{c}$.

(14) If the central attraction be

$$\mu\{2(a^2+b^2)u^5 - 3a^2b^2u^7\},$$

and the body be projected as in the last example, prove that the orbit will be the pedal of the ellipse with respect to the centre.

(15) A particle under a central attraction varying inversely as the fifth power of the distance is projected from a given point with a velocity which is to the velocity from infinity as 5 to 3, in a direction making an angle $\sin^{-1}\frac{2\sqrt{6}}{5}$ with the radius vector; find the orbit.

Here we have

$$\frac{d^2u}{d\theta^2}+u-\frac{\mu}{h^2}u^3=0;$$

$$\therefore h^2\left\{\left(\frac{du}{d\theta}\right)^2+u^2\right\}=v^2=C+\frac{\mu u^4}{2}.$$

But if V be the velocity of projection, c the initial value of u,

$$V^2=\frac{25}{9}\frac{\mu c^4}{2};$$

and when $$u=c,\ v=V,\quad \therefore\ C+\frac{8\mu c^4}{9};$$

$$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{\mu}{h^2}\left(\frac{8c^4}{9}+\frac{u^4}{2}\right).$$

But $$h^2=\frac{V^2\sin^2\beta}{c^2}=\frac{25\mu c^4}{18c^2}\frac{24}{25};$$

$$\therefore \frac{\mu}{h^2}=\frac{3}{4c^2}.$$

Substituting and integrating we find, after the necessary reductions,

$$r=\frac{\sqrt{3}}{2}R\frac{1-\epsilon^{\sqrt{2}(\theta-\alpha)}}{1+\epsilon^{\sqrt{2}(\theta-\alpha)}};$$

where R is the initial distance, and α a constant to be determined by the position of the initial line.

(16) If $P = 2\mu \dfrac{u^5}{c^2} + \mu u^3$, and a particle be projected at an angle of $\frac{1}{4}\pi$ with the initial distance $(R=)\dfrac{1}{c}$, with a velocity which is to the velocity in a circle at the same distance as $\sqrt{2}$ to $\sqrt{3}$, find the curve described.

$$r = R(1-\theta).$$

(17) A particle under a central attraction, varying partly as the inverse third, and partly as the inverse fifth, power of the distance, is projected with the velocity from infinity at an angle with the distance, the tangent of which is $\sqrt{2}$, the intensities being equal at the point of projection; determine the orbit.

$$R - r = \frac{1}{\sqrt{2}} R\theta.$$

(18) If $P = \dfrac{\mu}{r^5}(5r^2 - 8c^2)$, and a particle be projected from an apse at a distance c with the velocity from infinity; prove that the equation of the orbit is

$$r = \frac{1}{2} c\,(e^{2\theta} - e^{-2\theta}).$$

(19) If $P = 2\mu\left(\dfrac{1}{r^3} - \dfrac{a^2}{r^5}\right)$, and the particle be projected from an apse at a distance a with velocity $\dfrac{\sqrt{\mu}}{a}$, prove that it will be at a distance r after a time

$$\frac{1}{2\sqrt{\mu}}\left(a^2 \log \frac{r + \sqrt{r^2 - a^2}}{a} + r\sqrt{r^2 - a^2}\right).$$

(20) The attraction tending to the centre of a circle whose radius is a being $\mu\left(r + \dfrac{2a^3}{r^2}\right)$, find the velocity with which a particle will describe the circle; and shew that if the velocity be suddenly doubled the particle will come to an apse at the distance $3a$.

(21) If $P = \mu r + \frac{\nu}{r^3}$, prove that the equation of the orbit is of the form

$$\frac{1}{r^2} = \frac{\cos^2 k\theta}{a^2} + \frac{\sin^2 k\theta}{b^2}.$$

If the particle be projected from an apse at a distance $a = \sqrt[4]{\frac{\nu}{\mu}}$, with velocity $\sqrt[4]{\mu\nu}$, prove that the equation of the orbit is

$$r^2 = \frac{a^2}{1+\theta^2},$$

and that the time of describing the angle θ from the apse is

$$\frac{1}{\sqrt{\mu}} \tan^{-1} \theta.$$

(22) If a particle move under a central attraction $\mu u^2 + \nu u^3$, shew that the equation of the orbit is generally of the form

$$r = \frac{a}{1 - e\cos(k\theta)}.$$

In the case when the projection takes place at an apse, the apsidal distance being $\frac{\mu}{h}$, and ν being equal to h^2, shew that the equation of the path is

$$r = \frac{2\mu h^2}{2h^3 + \mu^2\theta^2},$$

and that the time of describing an angle α is

$$\frac{1}{\alpha} \tan\theta\,(\theta + \tfrac{1}{2}\sin 2\theta) \text{ where } \tan\theta = \frac{\mu\alpha}{\sqrt{(2h^3)}}.$$

Determine generally the relation between the orbits when $P = \mu u^2\phi(u)$ and when $P = \mu u^2\phi(u) + \nu u^3$.

(23) A particle is projected in any direction from one end of a uniform straight line each particle of which attracts it with an intensity proportional to the distance, prove that the particle will pass through the other end.

(24) A particle moves in an ellipse under an attraction tending to a fixed point O; prove that the acceleration due to the attraction at any point P varies as $\dfrac{DD'^6}{OP^2 . PP'^3}$, where PP' is the chord of the ellipse passing through O, and DD' the diameter parallel to PP'.

(25) A particle describes an equilateral hyperbola about a centre of attraction in the centre, shew that an angle θ from the apsidal line is connected with the time t of its description by the formula

$$\sin 2\theta = \frac{\epsilon^{4\sqrt{\mu}t} - 1}{\epsilon^{4\sqrt{\mu}t} + 1}.$$

(26) If v be the velocity of a particle moving in an ellipse about the centre, v' its velocity when the direction of its motion is at right angles to the former direction, the time of describing the intercepted arc $= \dfrac{1}{\sqrt{\mu}} \sin^{-1} \dfrac{vv'}{\mu ab}$.

(27) A particle moves under a central repulsion which varies as the distance from a fixed point; shew that the equation of the path described is

$$x\sqrt{y^2 - b^2} - y\sqrt{x^2 - a^2} = c,$$

where a, b, c are constants, and determine the curve which this equation represents.

(28) Find the time in which a particle would move from the vertex to the end of the latus rectum of a parabola, the centre of attraction being at the focus; and shew that if the velocity be there suddenly altered in the ratio m to 1 (m being < 1) the body will proceed to describe an ellipse, the excentricity of which is $(1 - 2m^2 + 2m^4)^{\frac{1}{2}}$.

(29) If the Earth's orbit be taken an exact circle, and a comet be supposed to describe round the Sun a parabolic orbit in the same plane; shew that the comet cannot possibly continue within the Earth's orbit longer than the $\left(\dfrac{2}{3\pi}\right)^{\text{th}}$ part of a year.

(30) If a particle, under a central attraction varying inversely as the square of the distance, be projected with a velocity equal to n times the velocity in a circle at the same distance; the angle α between the major axis and this distance may be determined from the equation

$$\tan(\alpha - \beta) = (1 - n^2) \tan \beta,$$

β being the angle between the radius vector and the direction of projection.

(31) A particle describes a parabola about a centre of attraction ($\propto D^{-2}$) residing in a point in the circumference of a given ellipse, the foci of which are in the circumference of the parabola; shew that the time of moving from one focus to the other is the same, at whatever point in the circumference of the ellipse the centre of attraction is placed.

(32) A particle is projected from a given point with a given velocity and is under a central attraction varying inversely as the square of the distance; shew that whatever be the direction of projection the centre of the orbit described will lie on the surface of a certain sphere.

(33) A particle revolves in a circle about a centre of attraction in the centre, the intensity $\propto \frac{1}{D^2}$; the strength is suddenly increased in the ratio of $m : 1$ when the particle is at any assigned point of its path, and when the particle arrives again at the same point the strength is again increased in the same ratio; shew that the path which the particle will describe is an ellipse whose excentricity

$$= \frac{m^2 - 1}{m^2}.$$

(34) A particle is moving in an ellipse about a centre of attraction in the focus; supposing that every time the particle arrives at the nearer apse the strength is diminished in the

ratio of 1 to $1-n$, find the excentricity of the elliptic orbit after p revolutions, the original excentricity being e.

$$\frac{1+e}{(1-n)^p}-1.$$

(35) If the attraction vary inversely as the square of the distance, prove that there are two initial directions in which a particle can move so that its apse line may coincide with a given line. If α_1, α_2 be the angles which these directions make with the initial distance c, and $2a$ be the length of the apse line, prove that

$$\cot\alpha_1 . \cot\alpha_2 = \frac{c}{a}-1.$$

(36) If the perihelion distance of a comet's orbit be $\frac{1}{8}$ of the radius of the Earth's orbit supposed circular, find the number of days the comet will remain within the Earth's orbit.

(37) If a comet describe 90° from perihelion in 100 days, compare its perihelion distance with the distance of a planet which describes its circular orbit in 942 days.

(38) In the case of planets and comets prove the following formulæ, the letters being the same as in the text,

$$r\frac{d\theta}{du}=a\sqrt{(1-e^2)};$$

$$\frac{r}{a}\sin\theta=\frac{\sqrt{(1-e^2)}}{e}(u-nt);$$

$$\log\frac{r}{a}=-\log(1+\lambda^2)$$
$$-2\left(\lambda\cos u+\tfrac{1}{2}\lambda^2\cos 2u+\tfrac{1}{3}\lambda^3\cos 3u+\&c.\right).$$

(39) A body describes an ellipse about the focus: prove that the times of describing the two parts, into which the orbit is divided by the minor axis, are to one another as $\pi+2e$ to $\pi-2e$, where e is the excentricity of the ellipse.

(40) If Pp, Qq be chords parallel to the major axis of an elliptic orbit, shew that the difference of the times through the arcs PQ, pq varies as the distance between the chords.

(41) If a comet whose orbit is inclined to the plane of the ecliptic were observed to pass over the Sun's disc, and three months after to strike the planet Mars, determine its distance from the Earth at the first observation, the Earth and Mars describing about the Sun circles in the same plane whose radii are as 2 : 3.

(42) Shew that the arithmetic mean of the distances of a planet from the Sun, at equal indefinitely small intervals of time, is

$$a\left(1+\frac{e^2}{2}\right).$$

(43) The time through an arc of a parabolic orbit bounded by a focal chord $\propto$ (chord)$^{\frac{3}{2}}$.

(44) If a circle be described passing through the focus and vertex of a parabolic orbit, and also through the position of the moving particle at each instant, shew that its centre describes with constant velocity a straight line bisecting at right angles the perihelion distance.

(45) Shew that the velocity of a comet perpendicular to the major axis varies inversely as its radius vector.

(46) D_1, D_2 being two distances of a comet, on opposite sides of perihelion, including a known angle, shew that the position of perihelion may be found from the equation

$$\frac{\sqrt{D_1}-\sqrt{D_2}}{\sqrt{D_1}+\sqrt{D_2}} = \tan \tfrac{1}{4} \text{ (sum of true anomalies)} \,.\, \tan \tfrac{1}{4} \text{ (difference)}.$$

(47) In an elliptic orbit find the relation between the mean angular velocity about the centre of attraction and the angular velocity about the other focus, and thence shew that when e is small the latter is nearly constant.

(48) If α, β be the greatest and least angular velocities in an ellipse about the focus, the *mean* angular velocity is

$$\frac{2\sqrt[4]{(\alpha^3\beta^3)}}{\sqrt{\alpha}+\sqrt{\beta}}.$$

(49) Find the maximum value of $\theta - nt$ in an elliptic orbit, and develop it in powers of e, shewing that it cannot contain even powers.

If Θ be this quantity,

$$\Theta = 2e + \frac{11e^3}{3 \cdot 2^4} + \frac{599e^5}{5 \cdot 2^{10}} + \&c.$$

(50) If $P = \mu u^3 (1 + k^2 \sin^2 \theta)^{-\frac{3}{2}}$, find the orbit, and interpret the result geometrically.

Find the equation of the orbit generally when $P = \mu u^3 f(\theta)$.

(51) Shew that if the central repulsion be constant ($=f$, suppose) we have the following relation between the radius vector and the time,

$$t = \int \frac{rdr}{\sqrt{2fr^2(r+a) - h^2}};$$

and from this, with the help of the equation of constant moment of momentum, deduce the differential equation of the orbit. Shew also how the apsidal angle may be determined.

If a particle, under a constant central repulsion, be projected from an apse with the velocity acquired from the centre, find the orbit.

(52) A particle moves about a centre of attraction, and its velocity at any point is inversely proportional to the distance from the centre of attraction; shew that its path will be a logarithmic spiral.

(53) Shew that the only law of central attraction for which the velocity at each point of the orbit can be equal to that in a circle at the same distance is that of the inverse third power, and that the orbit is the logarithmic spiral.

(54) If a number of particles, describing different circles in the same plane about a centre of attraction $\propto D^{-3}$, start together from the same radius, find the curve in which they all lie when that which moves in the circle whose radius is a has completed a revolution.

(55) If v be the velocity, and P the attraction at distance r in a central orbit, and if v', P', r' be similar quantities for the corresponding point of the locus of the foot of the perpendicular on the tangent, shew that

$$\frac{v^2}{Pr} + \frac{P'r'}{v'^2} = 2.$$

(56) A particle attached to one end of an elastic string moves on a smooth horizontal plane, the other end of the string being fixed to a point in the plane. If the path of the particle be a circle, shew that the periodic time $\propto \left(\frac{ra}{r-a}\right)^{\frac{1}{2}}$, a and r being the natural and stretched lengths of the string. If the orbit be nearly circular, find the apsidal angle.

(57) A particle is describing a curve about a centre of attraction, and its velocity $\propto \frac{1}{r^n}$, find the law of attraction and the equation of the path.

$$P \propto \frac{1}{r^{2n+1}}, \quad \left(\frac{r}{a}\right)^{n-1} = \cos(n-1)(\theta - \alpha).$$

(58) A particle projected in a given direction with a given velocity and attracted towards a given centre has its velocity at every point to the velocity in a circle at the same distance as 1 to $\sqrt{2}$; find the orbit described, the position of the apse, and the law of attraction.

$$r = \sqrt{\frac{\mu}{2h^2}\cos(\theta - \alpha)}, \quad P = \frac{\mu}{r^5}.$$

(59) If a particle move in a circle of radius r, about a centre of attraction distant a from the centre of the circle,

shew that the time from distance r to the nearer apse is

$$\frac{2^{\frac{3}{2}}r^{\frac{1}{2}}}{\sqrt{\phi}\left(2-\frac{a^2}{r^2}\right)^{\frac{3}{2}}}\left\{\cos^{-1}\frac{a}{2r}-\frac{a}{r}\sqrt{\left(1-\frac{a^2}{4r^2}\right)}\right\},$$

where ϕ is the initial attraction; and that the periodic time is

$$\frac{2\pi r^{\frac{3}{2}}}{(r-a)\sqrt{\phi}},$$

where ϕ is the attraction at the nearer apse.

(60) If the m^{th} power of the periodic time be proportional to the n^{th} power of the velocity in a circle, find the law of attraction in terms of the radius.

(61) A particle is projected at a distance c from a fixed centre of attraction with a velocity $\sqrt{\frac{\mu}{8c^4}}$, and in a direction making an angle $\sin^{-1}\frac{c}{a}$ with the distance; the intensity of the attraction at the distance r being $\frac{\mu r}{(r^2+c^2)^3}$. Shew that the orbit described will be a circle, of radius a.

(62) A point describes a parabola, latus rectum $4a$, with an acceleration tending to a point in the axis distant c from the vertex: prove that the time of moving from the vertex to a point distant y from the axis is proportional to $\frac{y^3}{12ac}+y$.

(63) If a body describes a parabola under an attraction tending to a point O on the axis, prove that the acceleration at any point P is $\mu\left(\frac{1}{OP}+\frac{1}{Op}\right)^{-2}OP^{-2}$, p being the point of intersection of PO produced with the curve.

Also prove that the time of passing from one end of the ordinate through O to the other $=\frac{8}{3}\sqrt{\frac{2}{\mu}}$.

(64) A particle P describes a cycloid ABC under an attraction tending to O the middle point of the base. If PM be drawn perpendicular to the axis OB, and PT the tangent meet OB in T: the angular velocity of the tangent will vary as $OM \,.\, OT$ inversely.

(65) If r, p be the radius vector and perpendicular on the tangent at any point of the curve described by a particle under an attraction P towards the pole, and a force T along the tangent, shew that

$$\frac{2Tp^2r}{\sqrt{r^2-p^2}} = \frac{d}{dr}\left(p^3 P \frac{dr}{dp}\right).$$

For an attraction P to the pole, and a force N in the normal, prove that

$$p^2 \frac{d}{dr}\left(Nr \frac{dr}{dp}\right) + \frac{d}{dr}\left(Pp^3 \frac{dr}{dp}\right) = 0.$$

(66) A particle describes the nth pedal freely under an attraction tending to a pole: find the law of attraction. If the curve be a rectangular hyperbola, and the pedals be formed with respect to its centre, prove that the nth pedal will be the orbit of a particle moving under an attraction varying as $r^{-\frac{6n+1}{2n-1}}$, where r is the distance from the centre of attraction.

(67) A particle describes an orbit round a centre of attraction in a periodic time P. Straight lines are drawn from a point to represent the accelerations of the particle at equal intervals of time τ, during a complete revolution. If $P = n\tau$, when n is an indefinitely great whole number, shew that these straight lines will represent a system of forces in equilibrium. Shew also that if the attraction vary directly as the distance, the result is true if n be not great.

(68) A particle describes an orbit about a centre of attraction. If the centre of attraction be replaced by the particle, and the orbit for any complete number of revolutions by a fine wire whose section varies inversely as the velocity in the corresponding orbit, and every point of which attracts

by the same law as the centre of attraction did, shew that the particle will be in equilibrium: determine also the nature of this equilibrium (1) when the attraction varies as the distance, (2) when it varies inversely as the square of the distance.

Shew that if the orbit be an ellipse, described about a centre of attraction in the focus, the centre of mass of the wire is midway between the centre and the other focus.

(69) If a uniform string under a central repulsion P per unit of length assume the form of a certain curve, prove that the same curve will be described by a particle of unit mass under a central attraction PT, the velocity at any point being numerically equal to the tension T of the string.

(70) If $P = \dfrac{\mu r}{(r^2 - c^2)^2}$, and if the particle be projected from an apse at a distance nc $(n > 1)$ with velocity which is to that in a circle as $\sqrt{n^2 - 1} : n$, prove that it will describe a branch of an epicycloid, and find the time to a cusp.

(71) Shew that if an ellipse be described under an attraction f to the focus S, and an attraction f' to the focus H, and $SP = r$, $HP = r'$,

$$\frac{df'}{dr'} - \frac{df}{dr} = 2\left(\frac{f}{r} - \frac{f'}{r'}\right).$$

(72) Prove that if $f = \mu \dfrac{r^3 + 8a^3}{8a^3 r^2}$, $f' = \mu \dfrac{r'^3 + 8a^3}{8a^3 r'^2}$, the ellipse can be described freely, and that the velocity at any point will be $n \dfrac{r^2 + rr' + r'^2}{2\sqrt{rr'}}$, n being the mean motion in the ellipse under an attraction $\dfrac{\mu}{r^2}$ to a focus.

(73) A particle describes an ellipse under two attractions tending to the foci which are to one another at any point inversely as the focal distances: prove that the velocity

varies as the perpendicular from the centre on the tangent, and that the periodic time $=\frac{\pi}{k}\left(\frac{a}{b}+\frac{b}{a}\right)$, ka, kb being the velocities at the ends of the axes.

(74) Prove that a particle can describe a parabola under a repulsion in the focus varying as the distance, and another force parallel to the axis always of three times the magnitude of the repulsion; and that if two equal particles describe the same parabola under these forces, their directions of motion will always intersect in a fixed confocal parabola.

(75) Prove that a lemniscate can be described freely by a particle under two central attractions of equal strength to the foci each varying inversely as the distance; and that the velocity will be always equal to $\sqrt{\frac{4\mu}{3}}$, μ being the strength of each attraction.

(76) If a particle move under an attraction μr to the point S, and a repulsion $\mu' r'$ from the point S', prove that

$$\mu r^2 \frac{d\theta}{dt} + \mu' r'^2 \frac{d\theta'}{dt} = c^2,$$

a constant, where θ, θ' are the angles r, r' make with SS'.

(77) The velocity of a point is the resultant of the velocities v and v' along radii-vectores r and r' measured from two fixed points at a distance a apart. Prove that the corresponding accelerations are

$$\frac{dv}{dt} + \frac{vv'}{2r^2 r'}(r^2 - r'^2 + a^2),$$

and

$$\frac{dv'}{dt} + \frac{vv'}{2rr'^2}(r'^2 - r^2 + a^2).$$

(78) A particle describes a circular orbit about a centre of attraction situated in the centre of the circle; prove that the form of the orbit will be stable or unstable according as the value of $\frac{d\log P}{d\log u}$, for $u=\frac{1}{a}$, is less or not less than 3, P

being the central attraction, u the reciprocal of the radius vector, and a the radius of the circle.

(79) If the equation for determining the apsidal distances in a central orbit contain the factor $(u-a)^p$, shew that $u=a$ cannot correspond to an apse unless p be of one of the forms $4m+2$ or $\dfrac{4m+2}{2n+1}$. If the factor $u-a$ occur twice, then a will be a root of the equation

$$\phi(u) - h^2u^3 = 0,$$

where $\phi(u)$ is the central attraction.

(80) Examine carefully the case of an apse where the centre of attraction coincides with the centre of curvature. Shew that the particle will, after passing such an apse, describe a circle about the centre of attraction, but that the motion will be unstable.

(81) A particle is projected from an apse under the attraction $\dfrac{f(r)}{r^3}$ with a velocity $\dfrac{(1+n)\sqrt{f(a)}}{a}$, n being very small and a the initial distance, determine the apsidal angle and the other apsidal distance.

(82) A particle moving in an ellipse about the focus is under a central disturbance which varies as $\dfrac{1}{r^2}\cos k\theta$, where θ is the longitude measured from the nearer apse, and k is nearly unity. Prove that in one revolution the apse line turns through an angle α, given by

$$(2\pi + \alpha)\cot\alpha = \text{constant}.$$

CHAPTER VI.

CONSTRAINED MOTION.

169. WE come now to the case of the motion of a particle subject not only to given forces, but to undetermined reactions. This occurs when the particle is attached to a fixed, or moving, point by means of a rod or string, and when it is forced to move on a curve or surface.

In applying to a problem of this kind the general equations of motion of a free particle, we must assume directions and intensities for the unknown reactions, treating them then as known, and it will always be found that the geometrical circumstances of the motion will furnish the requisite number of additional equations for the determination of all the unknown quantities in terms of the time, and the position of the particle.

One case of this kind has been already treated of (§ 84), namely, that of a particle moving on an inclined plane under gravity. There the undetermined reaction is the pressure on the plane, which however is evidently constant, and equal to the resolved part of the particle's weight perpendicular to the plane.

The laws of kinetic friction are but imperfectly known, and the few investigations which will be given of motion on a rough curve or surface are of very slight importance.

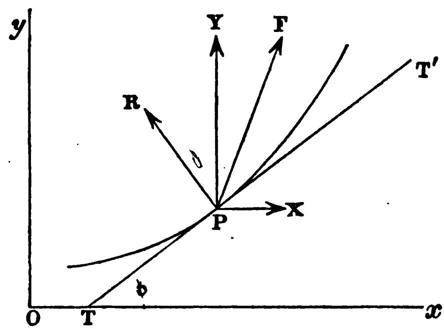
170. The simplest case is

A particle is constrained to move on a given smooth plane curve, under given forces in the plane of the curve, to determine the motion.

Taking rectangular axes in this plane, the forces may be resolved into two, X, Y, parallel respectively to the axes of x and y, the mass of the particle being taken as unity. In

addition there will be R, the pressure between the curve and particle, which acts in the normal to the curve, since the curve is smooth and there is therefore no friction.

Let P be the position of the particle at the time t; and let



the forces X, Y, R, act on the particle as in the figure, R being estimated positive towards the centre of curvature. Draw TP, a tangent to the constraining curve at P. Then the direction cosines of TP are

$$\frac{dx}{ds}, \frac{dy}{ds},$$

and those of PR are

$$\rho\frac{d^2x}{ds^2}, \rho\frac{d^2y}{ds^2}.$$

The equations of motion are

$$\frac{d^2x}{dt^2} = X + R\rho\frac{d^2x}{ds^2}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{d^2y}{dt^2} = Y + R\rho\frac{d^2y}{ds^2}\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

These two equations, together with the equation of the given curve, are sufficient to determine the motion completely.

To eliminate R, multiply (1) by $\frac{dx}{dt}$, (2) by $\frac{dy}{dt}$, and add. We thus obtain,

$$\text{since}\quad \frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\frac{d^2y}{ds^2}=0,$$

$$\frac{dx}{dt}\frac{d^2x}{dt^2}+\frac{dy}{dt}\frac{d^2y}{dt^2}=\frac{ds}{dt}\frac{d^2s}{dt^2}=X\frac{dx}{dt}+Y\frac{dy}{dt}\;\dots\dots\dots(3),$$

or, as we may write it,

$$\frac{d^2s}{dt^2}=X\frac{dx}{ds}+Y\frac{dy}{ds},$$

which might at once have been obtained by resolving along the tangent.

Now, it has been shewn in Chap. II. that if the forces resolved into X and Y are such as occur in nature,

$$Xdx+Ydy$$

is the complete differential of some function $-\phi(x,y)$.

Integrating (3) on this hypothesis, we have

$$\frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}=\frac{1}{2}v^2=C-\phi(x,y)\;\dots\dots(4),$$

supposing v to represent the velocity of the particle at the point xy.

Suppose the particle to start at the time $t=0$, from a point whose co-ordinates are a, b, with a velocity V.

We have, from (4),

$$\frac{1}{2}V^2=C-\phi(a,b);$$

and therefore $$\frac{1}{2}v^2=\frac{1}{2}V^2+\phi(a,b)-\phi(x,y)\;\dots\dots(5).$$

This shews that a particle, constrained to move under the forces X, Y, along any path whatever from the point a, b to the point x, y, has, on arriving at the latter point, the kinetic

energy increased by a quantity entirely independent of the path pursued: another simple case of the conservation of energy.

171. *To find the reaction of the constraining curve.*

Resolving along the normal PR, towards the centre of curvature,

$$\frac{v^2}{\rho} = R + X\rho\frac{d^2x}{ds^2} + Y\rho\frac{d^2y}{ds^2},$$

or

$$R = \frac{v^2}{\rho} - X\rho\frac{d^2x}{ds^2} - Y\rho\frac{d^2y}{ds^2},$$

which may also be written

$$R = \frac{v^2}{\rho} + X\frac{dy}{ds} - Y\frac{dx}{ds}.$$

This might, of course, have been obtained from (1) and (2) above, by multiplying them respectively by $\rho\frac{d^2x}{ds^2}$ and $\rho\frac{d^2y}{ds^2}$, and adding.

172. *To find the point where the particle will leave the constraining curve.*

For this it is evident that we have only to put $R = 0$, as then the motion will be free.

This condition gives

$$\frac{v^2}{\rho} = X\rho\frac{d^2x}{ds^2} + Y\rho\frac{d^2y}{ds^2}$$
$$= F\cos FPR,$$

if F be the resultant of X and Y.

Hence

$$\frac{1}{2}v^2 = F\frac{1}{2}\rho\cos FPR$$
$$= F\frac{1}{4}Q,$$

where Q is the chord of curvature in the direction PF.

Comparing this with the formula $\frac{1}{2}v^2 = fs$ (§ 82), we see that *the particle will leave the curve at a point where its velocity is such as would be produced by the resultant force then acting on it, if continued constant during its fall from rest through a space equal to* $\frac{1}{4}$ *of the chord of curvature parallel to that resultant.* (Compare § 144.)

This result is, from the analytical point of view, of little importance; but it is of great interest in connection with Newton's mode of treating such questions.

173. The formulæ just given are much simplified when we consider gravity only to be acting. Taking in this case the axis of y vertically *upwards*, our forces become

$$X = 0 \text{ and } Y = -g\,;$$

and the velocity, and the pressure on the curve, are given by

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}V^2 = g\,(k - y),$$

if $v = V$ when $y = k$;

and
$$\frac{v^2}{\rho} = R - g\frac{dx}{ds}.$$

Suppose we change the origin to the point from which the particle's motion is supposed to commence; and take the axis of y vertically *downwards;* we shall evidently have

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}V^2 = gy\,;$$

and if the particle starts from rest

$$\tfrac{1}{2}v^2 = gy.$$

This shews that the velocity depends merely on the distance beneath a horizontal plane through the original position of rest. Hence, whatever be the nature of the curve on which a particle slides under gravity, its motion will always be in the same direction till it rises to the same level as that to the fall from which its velocity is due. If it cannot do so, its motion will be constantly in the same direction; if it can, its velocity will become zero, and the particle will *then* either come permanently to rest, or return to the point from which it started.

174. *To find the time of a particle's sliding down any arc of a curve under gravity, from rest at the upper extremity of the arc.*

Taking the upper extremity as origin and the axis of y vertically downwards; we have

$$\frac{ds}{dt} = v = \sqrt{(2gy)};$$

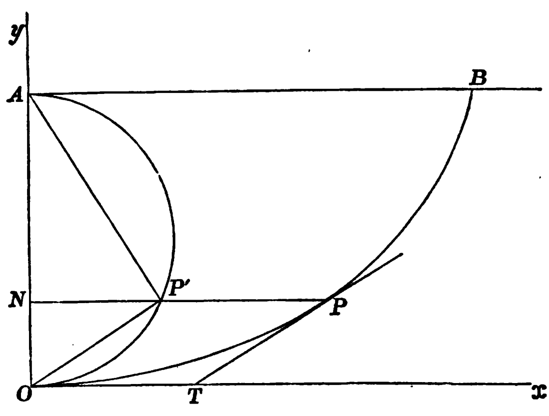
and
$$t_1 = \int_0^{y_1} \frac{\frac{ds}{dy}\,dy}{\sqrt{(2gy)}} \quad\text{..................(1)}$$

if y_1 be the vertical co-ordinate of the lower extremity of the given arc.

Or, taking the lower point as origin, and axis of y upwards, we have, since in this case v tends to decrease s,

$$t_1 = \int_{y_1}^{0} \frac{-\frac{ds}{dy}\,dy}{\sqrt{\{2g(y_1 - y)\}}} = \int_0^{y_1} \frac{\frac{ds}{dy}\,dy}{\sqrt{\{2g(y_1 - y)\}}} \quad\text{............(2).}$$

175. *To find the time of descending from rest at any point of an inverted cycloid to the vertex.*



Taking formula (2); since in this case the vertex is the

origin, and the axis is the axis of y, we have from the figure

$$s = OP = 2 \text{ chord } OP' = 2\sqrt{(AO \,.\, ON)} = 2\sqrt{(2ay)},$$

if a be the radius of the generating circle.

Hence, $$\frac{ds}{dy} = \sqrt{\frac{2a}{y}};$$

and $$t_1 = \sqrt{\frac{a}{g}}\int_0^{y_1} \frac{dy}{\sqrt{(yy_1 - y^2)}},$$

$$= \left(\sqrt{\frac{a}{g}} \operatorname{vers}^{-1} \frac{2y}{y_1}\right)_0^{y_1},$$

$$= \pi\sqrt{\frac{a}{g}};$$

which is independent of y_1, that is, of the point from which the particle begins its descent.

The reason of this remarkable property will be more easily seen if we take the formula for the acceleration in the direction of the arc. We have thus

$$\frac{d^2s}{dt^2} = -g \sin (P'Ox)$$

(since OP' is parallel to the tangent to the cycloid at P)

$$= -g \sin (OAP')$$

$$= -g\frac{OP'}{OA}$$

$$= -g\frac{s}{4a},$$

or the acceleration is proportional to the distance from the vertex measured along the cycloid.

176. *A particle, under gravity, moves in a vertical circle, to determine the motion.*

Taking the vertical diameter as axis of y, and its lower extremity as origin, the equation of the circle is

$$x = \sqrt{(2ay - y^2)}.$$

Hence $$\frac{ds}{dy}=\frac{a}{\sqrt{(2ay-y^2)}}.$$

But $$\frac{ds}{dt}=-\sqrt{\{2g(y_1-y)\}},$$

if we suppose the motion to be due to the level y_1 above the lowest point; and therefore

$$\frac{dt}{dy}=-\frac{a}{\sqrt{(2g)}}\frac{1}{\sqrt{\{(y_1-y)(2ay-y^2)\}}}\text{ ....(1).}$$

I. Suppose y_1 less than $2a$, the particle will then oscillate, and we must put $y=y_1\sin^2\phi$, and then

$$t=\sqrt{\frac{a}{g}}\int_0^\phi\frac{d\phi}{\sqrt{\left(1-\frac{y_1}{2a}\sin^2\phi\right)}}=\sqrt{\frac{a}{g}}F(k,\phi),\ k^2=\frac{y_1}{2a};$$

an elliptic integral of the first kind, of which ϕ is the amplitude and k the modulus.

Instead of considering t as a function of ϕ, we must consider ϕ as a function of t given by this equation, and then with Jacobi's notation put

$$\phi=\operatorname{am}\sqrt{\frac{g}{a}}t,$$

and therefore $$y=y_1\operatorname{sn}^2\sqrt{\frac{g}{a}}t,$$

and the time of vibrating from rest to rest is therefore $2K\sqrt{\frac{a}{g}}$, where K is the complete elliptic integral

$$\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}.$$

If the oscillations are indefinitely small, $k=0$ and $K=\frac{1}{2}\pi$, and the time of vibration from rest to rest is $\pi\sqrt{\frac{a}{g}}$, and therefore the time of a complete oscillation is $2\pi\sqrt{\frac{a}{g}}$.

II. Suppose y_1 greater than $2a$, the particle will then perform complete revolutions, and we must put

$$y = 2a \sin^2 \phi,$$

which gives

$$t = k\sqrt{\frac{a}{g}}\int_0^\phi \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}, \quad k^2 = \frac{2a}{y_1};$$

and therefore $$\phi = \text{am}\sqrt{\frac{g}{a}}\frac{t}{k},$$

and $$y = 2a \,\text{sn}^2 \sqrt{\frac{g}{a}}\frac{t}{k};$$

and the time of a complete revolution is

$$2\sqrt{\frac{a}{g}}Kk.$$

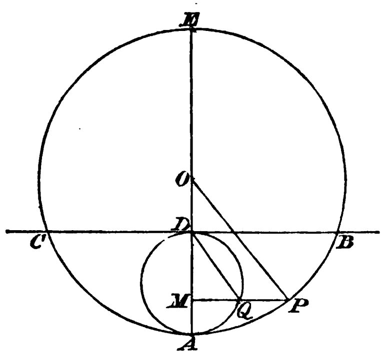
When the particle is supposed to be suspended by a thread without weight, it becomes what is termed a *simple pendulum*. Such a machine can exist only in theory, but Dynamics furnishes us with the means of reducing the calculation of the motion of such a pendulum as we can construct, to that of the simple pendulum. It is evident that by its means we may determine the value of g, if the length of the pendulum, its arc of oscillation, and the number of vibrations it makes in a given time, be known. Since gravity decreases (according to a known law) as we ascend above the Earth's surface, the comparison of the times of vibration of the same pendulum on the top of a mountain and at its base would give approximately the height. Similarly, the comparison of the times of vibration above ground, and at the bottom of a coal-pit, gives information as to the Mean Density of the Earth. One of the most important applications of the pendulum is that made by Newton. It is evident that if the weight of a body be not proportional to its mass, the value of g will be different for different materials. Hence the fact that pendulums of the same length vibrate in equal times at the same place whatever be the matter of which the bob is made, proves, by means of the above formula, the truth of

one part of the Law of Gravitation: viz. that, ceteris paribus, the attraction exerted by one body on another is proportional to the quantity of matter it contains, and independent of its quality.

177. We may determine the motion of the simple circular pendulum by resolving along the arc. The details of the process will shew the nature of the Elliptic Function transformations.

Let O be the centre of the circle, OA the vertical radius, P the position of the particle at the time t, and let $AOP=\theta$.

Suppose the motion to be due to the level BC: then we must distinguish the two cases in which BC does and does not cut the circle.



I. Suppose BC to cut the circle in B and C, and let $AOB=\alpha$; then the pendulum will oscillate through an angle 2α.

The equation of motion will be

$$\frac{d^2s}{dt^2}=-g\sin\theta.$$

But $$s=a\theta,$$

therefore $$\frac{d^2\theta}{dt^2}=-\frac{g}{a}\sin\theta.$$

Multiplying by $\frac{d\theta}{dt}$ and integrating

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{g}{a}\left(\cos\theta - \cos\alpha\right) \qquad (1).$$

Let BC cut OA in D; on AD as diameter describe a circle, and let PM drawn perpendicular to OA cut this circle in Q, and let $ADQ = \phi$.

Then since $AM = a(1 - \cos\theta) = 2a\sin^2\frac{\theta}{2}$

and $$AM = AD\sin^2\phi = 2a\sin^2\frac{\alpha}{2}\sin^2\phi;$$

therefore $$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\phi.$$

Substituting in equation (1), we obtain

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{g}{a}\left(1 - \sin^2\frac{\alpha}{2}\sin^2\phi\right),$$

and $$t = \sqrt{\frac{a}{g}}\int_0^\phi \frac{d\phi}{\sqrt{(1 - k^2\sin^2\phi)}}, \quad k = \sin\frac{\alpha}{2}.$$

Therefore as before

$$\phi = \operatorname{am}\left(\sqrt{\frac{g}{a}}\,t, k\right),$$

$$\sin\frac{\theta}{2} = k\operatorname{sn}\sqrt{\frac{g}{a}}\,t,$$

$$\cos\frac{\theta}{2} = \operatorname{dn}\sqrt{\frac{g}{a}}\,t;$$

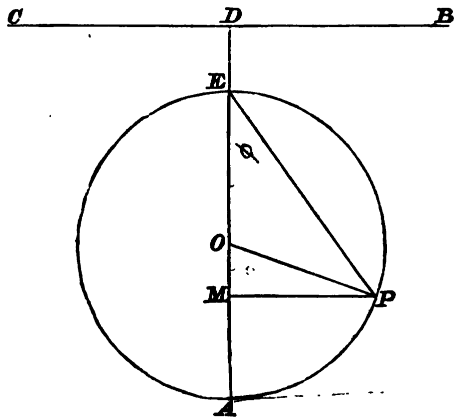
therefore $$AP = AB\operatorname{sn}\sqrt{\frac{g}{a}}\,t,$$

$$EP = EA\operatorname{dn}\sqrt{\frac{g}{a}}\,t,$$

$$\frac{d\theta}{dt} = 2k\sqrt{\frac{g}{a}}\,\mathrm{cn}\sqrt{\frac{g}{a}}\,t,$$

and $$AM = AD\,\mathrm{sn}^2\sqrt{\frac{g}{a}}\,t,$$

as before.



II. Suppose BC not to cut the circle, then the pendulum will perform complete revolutions; let $AD = y_1$.

Then, as before, resolving along the tangent

$$\frac{d^2\theta}{dt^2} = -\frac{g}{a}\sin\theta,$$

and $$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = C + \frac{g}{a}\cos\theta;$$

and when $\theta = 0$, $$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{gy_1}{a^2};$$

therefore $$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 = \frac{gy_1}{a^2} - \frac{g}{a}(1-\cos\theta).$$

Let $AEP = \phi$, therefore $\theta = 2\phi$,

and $$\left(\frac{d\phi}{dt}\right)^2 = \frac{gy_1}{2a^2} - \frac{g}{a}\sin^2\phi$$

$$= \frac{gy_1}{2a^2}\left(1 - \frac{2a}{y_1}\sin^2\phi\right)$$

$$= \frac{g}{ak^2}(1 - k^2\sin^2\phi),\ k^2 = \frac{2a}{y_1}.$$

Therefore $$\phi = \operatorname{am}\left(\sqrt{\frac{g}{a}}\frac{t}{k}, k\right),$$

and $$AM = 2a\,\operatorname{sn}^2\sqrt{\frac{g}{a}}\frac{t}{k},$$

as before.

In the separating case BC touches the circle at its highest point E, and $y_1 = 2a$, $k = 1$; therefore

$$\frac{d\phi}{dt} = \sqrt{\frac{g}{a}}\cos\phi,$$

$$t = \sqrt{\frac{a}{g}}\int_0^\phi \frac{d\phi}{\cos\phi}$$

$$= \sqrt{\frac{a}{g}}\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right),$$

$$\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = e^{\sqrt{\frac{g}{a}}t},$$

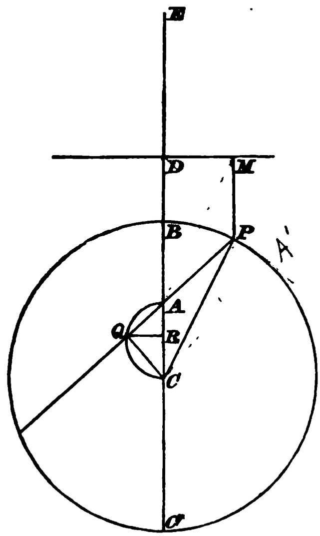
$$\frac{1 + \sin\phi}{1 - \sin\phi} = e^{2\sqrt{\frac{g}{a}}t},$$

$$\sin\phi = \frac{e^{\sqrt{\frac{g}{a}}t} - e^{-\sqrt{\frac{g}{a}}t}}{e^{\sqrt{\frac{g}{a}}t} + e^{-\sqrt{\frac{g}{a}}t}},$$

which determines the motion completely.

Since $\phi = \frac{1}{2}\pi$ when $t = \infty$, the pendulum will continually approach the highest position, but never reach it.

178. Let DM be a horizontal line, and let DA be taken equal to the tangent from D to the circle BPC', whose centre C is vertically under D. Also let PAQ be any line through A,



cutting in Q the semicircle on AC. Also make E the image of A in DM. Then if P move with velocity due to the level of DM, Q moves with velocity due to the level of E; so that we have the means of comparing, arc for arc, two different continuous forms of pendulum motion, in one of which the rotation takes place in half the time of that in the other.

Let ω be a small increment of the circular measure of BAP, then arc at $Q = CA \,.\, \omega$, arc at $P = \dfrac{AP \,.\, PC}{PQ} \,.\, \omega$.

Also,

$$\text{velocity at } P = \sqrt{2g \,.\, PM} = \sqrt{\frac{g}{AC}} \,.\, AP.$$

Hence,

$$\text{velocity at } Q = \frac{CA \,.\, PQ}{AP \,.\, PC}\sqrt{\frac{g}{AC}} \,.\, AP = \frac{\sqrt{g \,.\, AC}}{PC} \,.\, PQ.$$

But

$$PQ = \sqrt{CP^2 - CQ^2}$$

$$= \sqrt{CP^2 - CR \,.\, CA} \text{ (where } QR \text{ is horizontal)}$$

$$= \sqrt{CA}\sqrt{\frac{CP^2 - CA^2}{CA} + AR} = \sqrt{CA \,.\, ER}.$$

Hence,

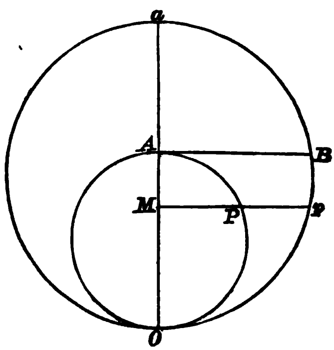
$$\text{velocity at } Q = \frac{AC}{PC}\sqrt{g \,.\, ER}.$$

Thus Q moves with velocity due to the level of E, and constant acceleration

$$\frac{AC^2}{2PC^2} \cdot g.$$

We have at once the means of comparing continuous rotation with oscillation, as follows:

Let two circles touch one another at their lowest points; compare the arcual motions of points P and p, which are



always in the same horizontal line. Draw the horizontal tangent AB. Then, if the line MPp be slightly displaced,

$$\frac{\text{arc at } P}{\text{arc at } p} = \frac{AO}{MP} \cdot \frac{Mp}{aO} = \frac{AO}{aO}\sqrt{\frac{aM \,.\, MO}{AM \,.\, MO}} = \frac{AO}{aO}\sqrt{\frac{aM}{AM}}.$$

Thus, if P move, with velocity due to g and level a, continuously in its circle, p oscillates with velocity due to

$$g \cdot \frac{aO^2}{AO^2} \text{ and level } AB.$$

Combining the two propositions, we are enabled to compare the times of oscillation in two different arcs of the same or of different circles.

It is obvious that the squares of the sines of the quarter arcs of vibration which the combination of the above processes leads us to compare are (in the first figure),

$$\frac{CA}{CE} \text{ and } \frac{C'B}{C'D} \text{ respectively.}$$

Calling them $\frac{1}{k^2}$ and $\frac{1}{k_1^2}$, and putting $DA = a$, $AC = e$, we have

$$\frac{1}{k^2} = \frac{e}{2a+e}, \quad \frac{1}{k_1^2} = \frac{2\sqrt{2ae+e^2}}{a+e+\sqrt{2ae+e^2}}.$$

Hence

$$\frac{1}{k_1^2} = \frac{\frac{4}{k}}{1+\frac{1}{k^2}+\frac{2}{k}},$$

or

$$\frac{1}{k_1} = \frac{2\sqrt{k}}{1+k}.$$

Lagrange's transformation is equivalent to

$$\sin\phi = \frac{2\sqrt{k}\sin\theta}{1+k\sin^2\theta},$$

which, for limits 0 and $\sin^{-1}\frac{1}{k}$ for θ, gives 0 and $\sin^{-1}\frac{1}{k_1}$ for ϕ;

and we thus have

$$\int_0^{\sin^{-1}\frac{1}{k_1}} \frac{d\phi}{\sqrt{\frac{1}{k_1^2} - \sin^2\phi}} = \frac{2k_1}{\sqrt{k}} \int_0^{\sin^{-1}\frac{1}{k}} \frac{d\theta}{\sqrt{\frac{1}{k^2} - \sin^2\theta}},$$

whose application to the pendulum problem is obvious.

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179. *To find the pressure between the circle and the particle, or the tension of the string.*

The reaction R being measured positively as a tension between the particle and the centre,

$$R = \frac{v^2}{a} + g\cos\theta.$$

In the figure of § 177, let $AD = y_1$, BC being the level to which the motion is due; then

$$\frac{1}{2}v^2 = g\{y_1 - a(1 - \cos\theta)\},$$

and therefore

$$R = g\left\{\frac{2y_1}{a} - 2(1 - \cos\theta) + \cos\theta\right\}$$

$$= 3g\left\{\cos\theta - \frac{2}{3}\left(1 - \frac{y_1}{a}\right)\right\}.$$

This expression for R admits of the value zero, if

$$\frac{2}{3}\left(1 - \frac{y_1}{a}\right) \nless -1, \text{ or } y_1 \ngtr \frac{5}{2}a.$$

It may happen however that when the particle oscillates, the points thus found may not lie within the arc which the particle passes over.

The particle will oscillate if $y_1 < 2a$. Now in order that the points where R vanishes may lie within the limits of

oscillation, the value of $\cos\theta$ for the former must not be less than that for the latter, and therefore

$$\frac{2}{3}\left(1-\frac{y_1}{a}\right) \nless 1-\frac{y_1}{a},$$

or

$$y_1 \nless a.$$

Hence, if $a < y_1 < \frac{5}{2}a$, there will be a point at which R vanishes; and if the particle be moving on the concave side of a smooth circle, or be attached by a string to a fixed point, the circular motion will cease at this point; the particle will fall off the circle in the one case, and the string will cease to be stretched in the other.

If, however, the particle be confined in a circular tube, or attached to the centre by a light rigid rod, the pressure on the particle will act outwards from the centre, or the stress in the rod will change from a tension to a pressure.

Beyond these limits it is evident we shall have complete revolutions if $y_1 > \frac{5}{2}a$, and oscillations if $y_1 < a$, without change of sign of R, or without the string becoming slack.

Also by what we have before shewn, if the particle be constrained by a circular tube or light rigid rod, it will oscillate if $y_1 < 2a$; if $y_1 = 2a$, the particle will reach the highest point after the lapse of an infinite time, and if $y_1 > 2a$, the particle will revolve continuously.

180. *Two points being given, which are neither in a vertical nor in a horizontal line, to find the curve joining them, down which a particle sliding under gravity, and starting from rest at the higher, will reach the other in the least possible time.*

The curve must evidently lie in the vertical plane passing through the points. For suppose it not to lie in that plane, project it orthogonally on the plane, and call corresponding elements of the curve and its projection σ and σ'. Then if a particle slide down the projected curve its velocity at σ' will

be the same as the velocity in the other at σ. But σ is *never less* than σ', and is generally greater. Hence the time through σ' is generally less than that through σ, and *never greater*. That is, the whole time of falling through the projected curve is less than that through the curve itself. Or the required curve lies in the vertical plane through the points.

Taking the axes of x and y, horizontal, and vertically downwards, respectively, from the starting point; if x_0 be the abscissa of the other point, the time of descent will be

$$t_0 = \int_0^{x_0} \frac{\frac{ds}{dx}dx}{\sqrt{(2gy)}}; \text{ or, writing } \frac{dy}{dx} = p,$$

$$t_0 = \int_0^{x_0} \frac{\sqrt{(1+p^2)}}{\sqrt{(2gy)}}\, dx.$$

Applying the rules of the Calculus of Variations, we have, since V or $\dfrac{\sqrt{(1+p^2)}}{\sqrt{y}}$ is a function of y and p, the condition for a minimum,

$$V = p\frac{dV}{dp} + C,$$

the differential coefficient being partial.

This gives $$\frac{\sqrt{(1+p^2)}}{\sqrt{y}} = \frac{p^2}{\sqrt{y}\sqrt{(1+p^2)}} + C,$$

or $$\sqrt{y}\sqrt{(1+p^2)} = \frac{1}{C} = \sqrt{a} \text{ suppose.}$$

Hence $$\frac{ds}{dy} = \frac{\sqrt{(1+p^2)}}{p} = \sqrt{\frac{a}{a-y}},$$

the differential equation of a cycloid, the origin being a cusp and the base the axis of x.

This is a problem celebrated in the history of Dynamics. The cycloid has received on account of this property the name

of Brachistochrone. Farther on we propose to investigate the nature and some of the properties of Brachistochrones for other forces besides gravity. For an investigation not directly involving the Calculus of Variations see Appendix.

181. *A particle moves on a smooth plane curve under an attraction to a fixed centre in the plane of the curve; to determine the motion.*

Let $r = f(\theta)$ be the polar equation of the constraining curve about the centre of force as pole, and let $P = \phi(r)$ be the attraction on a particle whose distance from the centre is r.

Resolving along the tangent at any point,

$$\frac{d^2s}{dt^2} = -P\frac{dr}{ds} \qquad (1).$$

Hence, $$\frac{1}{2}\left(\frac{ds}{dt}\right)^2 = \tfrac{1}{2}v^2 = C - \int\phi(r)\,dr \qquad (2).$$

Equation (2) contains the complete solution of the problem so far as the motion is concerned; since, by means of the equation of the curve, either r or s may be eliminated from it, and if the resulting differential equation be integrable, it will give s or r in terms of t.

For the pressure on the curve. Resolving along the normal at any point, ρ being the radius of curvature, we have

$$\frac{v^2}{\rho} = R + Pr\frac{d\theta}{ds} \qquad (3),$$

an expression which by means of the foregoing equations will give R in terms of t or r.

Hence the solution is complete.

182. *When the constraining curve is tortuous.*

All we know directly about R is that it is perpendicular to the tangent line at any point.

Resolve then the given forces acting upon the particle into three, one, T, along the tangent, which in all cases in nature will be a function of x, y, z and therefore of s; another, N, in the line of intersection of the normal and osculating planes (or radius of absolute curvature); and the third, P, perpendicular to the osculating plane.

Let the resolved parts of R in the directions of N and P be R_1, R_2. Now the acceleration of a point moving in any manner is compounded of two accelerations, one $\frac{d^2s}{dt^2}$ or $v\frac{dv}{ds}$ along the tangent to the path, and the other $\frac{v^2}{\rho}$ towards the centre of absolute curvature, the acceleration perpendicular to the osculating plane being zero; and therefore

$$\frac{d^2s}{dt^2} = T \text{.............................(1).}$$

This equation together with the two of the curve is sufficient to determine the *motion* completely.

Also $$\frac{v^2}{\rho} = R_1 + N \text{.........................(2),}$$

R_1 and T being considered positive when acting towards the centre of absolute curvature; this equation determines R_1.

Now R_2 is the reaction which prevents P's withdrawing the particle from the osculating plane; and therefore

$$R_2 = -P \text{...........................(3),}$$

(2) and (3) give the resolved parts of the pressure on the curve.

Also $R = \sqrt{(R_1^2 + R_2^2)}$, and its direction makes an angle $= \tan^{-1}\left(\frac{R_2}{R_1}\right)$ with the osculating plane.

183. In Art. 175 we arrived at the remarkable property of the inverted cycloid, that a particle falling under gravity from rest at any point of the curve reaches the lowest point

in the same time, whatever be the point of the curve from which it starts. *Let us find for what forces an analogous property is possessed by any other given curve*

Let the forces resolved along the curve have a component $= -\phi'(s)$, where s is the distance from the point to which the time of fall is constant: then,

$$\frac{d^2 s}{dt^2} = -\phi'(s) \qquad (1).$$

If the particle starts at a distance k from the fixed point, the velocity $= 0$ when $s = k$. Hence the corrected integral of (1) is

$$\frac{1}{2}\left(\frac{ds}{dt}\right)^2 = \phi(k) - \phi(s),$$

and we have
$$\sqrt{2}\,\tau = \int_0^k \frac{ds}{\{\phi(k) - \phi(s)\}^{\frac{1}{2}}};$$

if τ be the time of fall to the fixed point, which is by hypothesis to be independent of k.

Put $s = kz$, the limits of z are 1 and 0, and

$$\sqrt{2}\,\tau = \int_0^1 \frac{kdz}{\{\phi(k) - \phi(kz)\}^{\frac{1}{2}}};$$

and, that this may be independent of k, we must obviously have
$$\phi(k) - \phi(kz) = k^2 f(z).$$

This may be put in the form

$$\frac{\phi(k)}{k^2} - z^2\,\frac{\phi(kz)}{k^2 z^2} = f(z),$$

from which, by inspection, we obtain

$$\frac{\phi(k)}{k^2} = C' + \frac{C''}{k^2} \qquad (2).$$

Or we might have proceeded as follows.

Put $\frac{\phi k}{k^2} = \psi k$, then $\psi(k) - z^2\psi(kz) = fz$.

By differentiation with regard to k,

$$\psi'(k) - z^3\psi'(kz) = 0.$$

This shows that $k^3\psi'(k)$ is an absolute constant.

Hence, or by (2),

$$\phi'(s) = Cs.$$

Thus, by (1), $$\frac{d^2s}{dt^2} = -Cs \quad\text{.......................}(3),$$

that is, the resolved force along the curve must be proportional to the arcual distance from the fixed point.

Hence, if X, Y, Z be the impressed forces,

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = -Cs$$

is the condition they must satisfy at every point x, y, z of the given curve. For such forces the given curve is said to be a *Tautochrone.*

By § 90, the time of descent is

$$\tau = \frac{\pi}{2\sqrt{C}}. \quad \text{Hence } C = \frac{\pi^2}{4\tau^2}.$$

184. *To find the Brachistochrone for a particle subjected to any forces which make* $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ *a complete differential of three independent variables.*

Generally

$$t = \int\frac{ds}{v},$$

between proper limits, is to be a minimum; and therefore, taking its variation,

$$\delta t = \int\frac{v\delta ds - ds\delta v}{v^2} = 0 \quad\text{..................}(1).$$

But the equation of energy is

$$\frac{1}{2}v^2 = \int (Xdx + Ydy + Zdz);$$

and gives $\qquad v\delta v = X\delta x + Y\delta y + Z\delta z,$

or $\qquad ds\,\delta v = (X\delta x + Y\delta y + Z\delta z)\,dt \ldots\ldots\ldots\ldots\ldots (2).$

Again $\qquad ds^2 = dx^2 + dy^2 + dz^2,$

and $\quad \dfrac{ds}{dt}\delta ds = v\delta ds = \dfrac{dx}{dt}\delta dx + \dfrac{dy}{dt}\delta dy + \dfrac{dz}{dt}\delta dz \ldots\ldots\ldots\ldots (3).$

Hence (1) becomes, by (2) and (3), and since d and δ follow the commutative law,

$$0 = \int \frac{1}{v^2}\left(\frac{dx}{dt}d\delta x + \frac{dy}{dt}d\delta y + \frac{dz}{dt}d\delta z\right) - \int \frac{1}{v^2}(X\delta x + Y\delta y + Z\delta z)\,dt$$

$$= \left[\frac{1}{v^2}\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right)\right] - \left\{\frac{1}{v^2}\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right)\right\}$$

$$- \int dt\left[\left\{\frac{d}{dt}\left(\frac{1}{v^2}\frac{dx}{dt}\right) + \frac{X}{v^2}\right\}\delta x + \left\{\frac{d}{dt}\left(\frac{1}{v^2}\frac{dy}{dt}\right) + \frac{Y}{v^2}\right\}\delta y + \left\{\frac{d}{dt}\left(\frac{1}{v^2}\frac{dz}{dt}\right) + \frac{Z}{v^2}\right\}\delta z\right],$$

by integrating the first term by parts. The integrated terms in [] belong to the superior, those in {} to the inferior, limit.

But, if the terminal points are given, we have at both limits

$$\delta x = 0, \quad \delta y = 0, \quad \delta z = 0,$$

and therefore the terms independent of the integral sign vanish. In order that the integral may be identically zero, we must have, since δx, δy, δz are independent,

$$\frac{d}{dt}\left(\frac{1}{v^2}\frac{dx}{dt}\right) + \frac{X}{v^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

with similar expressions in y and z. The elimination of t, and v or $\frac{ds}{dt}$, from these equations will give us the two differential equations of the curve required, the forces X, Y, Z being by hypothesis functions of x, y, z only.

185. But without getting rid of v we may prove two properties common to all such Brachistochrones.

Eliminating t from (4) we have

$$v\frac{d}{ds}\left(\frac{1}{v}\frac{dx}{ds}\right)+\frac{X}{v^2}=0,$$

or

$$v^2\frac{d^2x}{ds^2}-v\frac{dv}{ds}\frac{dx}{ds}+X=0\ldots\ldots\ldots\ldots\ldots(5),$$

with similar expressions in y and z.

Multiplying these in order by λ, μ, ν and adding; if we take λ, μ, ν such that

$$\left.\begin{aligned}\lambda\frac{d^2x}{ds^2}+\mu\frac{d^2y}{ds^2}+\nu\frac{d^2z}{ds^2}&=0\\ \lambda\frac{dx}{ds}+\mu\frac{dy}{ds}+\nu\frac{dz}{ds}&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots(6),$$

we shall have also

$$\lambda X+\mu Y+\nu Z=0\ldots\ldots\ldots\ldots\ldots(7).$$

Now (6) shows that the line whose direction cosines are as λ, μ, ν is perpendicular to the radius of absolute curvature of the path, and also to the tangent; that is, it is normal to the osculating plane. Also by (7) the same line is perpendicular to the resultant of X, Y, Z.

Hence, *the osculating plane at any point contains the resultant of the impressed forces.*

Again, if ρ be the radius of absolute curvature,

$$\frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2,$$

and its direction cosines are

$$\rho\frac{d^2x}{ds^2},\ \rho\frac{d^2y}{ds^2},\ \rho\frac{d^2z}{ds^2};$$

therefore, multiplying equations (5) by

$$\frac{d^2x}{ds^2},\ \frac{d^2y}{ds^2},\ \frac{d^2z}{ds^2},$$

and adding; noting that, since

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1,$$

we have

$$\frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\frac{d^2y}{ds^2}+\frac{dz}{ds}\frac{d^2z}{ds^2}=0;$$

we obtain the equation

$$\frac{v^2}{\rho}=-\left(X\rho\frac{d^2x}{ds^2}+Y\rho\frac{d^2y}{ds^2}+Z\rho\frac{d^2z}{ds^2}\right)\ldots\ldots\ldots\ldots\ldots(8),$$

or the normal component of the impressed forces in a Brachistochrone is equal and opposite to the normal component of the forces which with the same velocity would cause the Brachistochrone to be described freely.

The velocity in the curve supposed to be a Brachistochrone or a free path being the same, the tangential component of the impressed forces must be the same, and therefore if we reflect the impressed force in the tangent at every point, the Brachistochrone becomes a free path, and vice versa; in this way from the known properties of free paths we can find for what forces they are Brachistochrones and conversely.

Thus from the properties of free parabolic or elliptic motion we obtain that, a parabola for a constant repulsion from the focus, or an ellipse for a repulsion from one focus inversely as the square of the distance from the other focus is a Brachistochrone, the circle of zero velocity being evanescent.

186. If the terminal points are not definitely assigned (if, for instance, it be required to find the line of swiftest descent from one given curve to another) we have no longer

$$\delta x = 0,\ \delta y = 0,\ \delta z = 0$$

at the limits; but, with the requisite modifications, the process in § 184 enables us to find the proper conditions in any case. Such questions, however, involve difficulties belonging rather to Calculus of Variations than to Kinetics.

Thus, suppose that the final point of the path is to lie on

$$F(x, y, z) = 0,$$

we have

$$\frac{dF}{dx}\delta x + \frac{dF}{dy}\delta y + \frac{dF}{dz}\delta z = 0 \text{ ....(1).}$$

Also that [ ] may vanish, which is necessary in order that δt may be zero, we must have

$$\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z = 0 \text{ ....(2).}$$

Now the only relation between δx, δy and δz is (1), to which (2) must therefore be equivalent: hence

$$\frac{dx}{dt} : \frac{dy}{dt} : \frac{dz}{dt} :: \frac{dF}{dx} : \frac{dF}{dy} : \frac{dF}{dz}.$$

These equations show that the moving particle meets the terminal surface at right angles. A similar condition is easily seen to hold if the initial point of the path is also to lie on a given surface, provided the whole energy be given and the given surface be an *equipotential* one. If it be not equipotential, terms depending on δx_0, δy_0, δz_0, will appear in the integral and must be taken along with { }.

If a terminal point is to lie on a given *curve* the condition is to be determined in a similar manner.

187. *A particle moves under given forces on a given smooth surface; to determine the motion, and the pressure on the surface.*

Let

$$F(x, y, z) = 0 \quad\dots\dots\dots\dots\dots\dots(1),$$

be the equation of the surface, R the reaction, acting in the normal to the surface, which is the only effect of the constraint. Then if λ, μ, ν be its direction cosines, we know that

$$\lambda = \frac{\left(\frac{dF}{dx}\right)}{\sqrt{\left\{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2\right\}}} \quad\dots\dots\dots\dots(2),$$

with similar expressions for μ and ν; the differential coefficients being partial.

If X, Y, Z be the impressed forces on unit of mass, our equations of motion are, evidently,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= X + R\lambda \\ \frac{d^2y}{dt^2} &= Y + R\mu \\ \frac{d^2z}{dt^2} &= Z + R\nu \end{aligned}\right\} \quad\dots\dots\dots\dots(3).$$

Multiplying equations (3) respectively by

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt},$$

and adding, we obtain

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2} = \frac{1}{2}\frac{d(v^2)}{dt}$$

$$= X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt} \quad\dots\dots\dots\dots(4).$$

R disappears from this equation, for its coefficient is

$$\lambda \frac{dx}{dt} + \mu \frac{dy}{dt} + \nu \frac{dz}{dt},$$

and vanishes, because the line whose direction cosines are proportional to $\frac{dx}{dt}$, &c. being the tangent to the path, is perpendicular to the normal to the surface.

If we suppose X, Y, Z to be forces such as occur in nature, (Chap. II.) the integral of (4) will be of the form

$$\tfrac{1}{2} v^2 = \phi(x, y, z) + C \quad\text{.................. (5)},$$

and the velocity at any point will depend only on the initial circumstances of projection, and not on the form of the path pursued.

To find R, resolving along the normal, then

$$\frac{v^2}{\rho} = X\lambda + Y\mu + Z\nu + R,$$

which gives the reaction of the surface; ρ being the radius of curvature of the normal section of the surface through the tangent to the path.

188. *To find the curve which the particle describes on the surface.*

For this purpose we must eliminate R from equations (3). By this process we obtain

$$\frac{\frac{d^2x}{dt^2} - X}{\lambda} = \frac{\frac{d^2y}{dt^2} - Y}{\mu} = \frac{\frac{d^2z}{dt^2} - Z}{\nu} \quad\text{.................. (6)},$$

two equations, between which if t be eliminated, the result is the differential equation of a second surface intersecting the first in the curve described.

If there be no impressed forces, or if the component of the impressed force in the tangent plane coincide with the direction of motion of the particle, then the osculating plane of the path of the particle, which contains the

resultant of R and the impressed force, will be a normal plane, and therefore the path will be a geodesic on the surface.

Thus a particle under no forces on a smooth (or rough) surface will describe a geodesic.

189. *A particle moves on a surface of revolution, under gravity acting in a direction parallel to the axis of the surface; to determine the motion.*

Take the axis of the surface as that of z, the equation may be written

$$F(x, y, z) = f\{\sqrt{(x^2 + y^2)}\} - z = 0.$$

This may be put in the form

$$f(\rho) - z = 0,$$

if ρ be the distance of any point in the surface from the axis.

Equations (6) become

$$\frac{\dfrac{d^2x}{dt^2}}{f'(\rho)\dfrac{x}{\rho}} = \frac{\dfrac{d^2y}{dt^2}}{f'(\rho)\dfrac{y}{\rho}} = \frac{\dfrac{d^2z}{dt^2} - g}{-1} \quad \text{...............(7).}$$

The first two equal terms give us, for the projection of the motion on a horizontal plane, the equation

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0.$$

But if θ be the angle between the plane containing ρ and the axis of z, and a fixed plane through that axis; we see that this is equivalent to

$$\rho^2\frac{d\theta}{dt} = \text{const.} = h \quad \text{.....................(8).}$$

If the motion be due to the level k, the second integral of equations (7) is

$$\frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = g(k - z).$$

Let $u = \frac{1}{\rho}$, and $z = \phi(u)$ be the equation of the surface; then

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\},$$

$$\frac{dz}{dt} = hu^2\,\phi'(u)\,\frac{du}{d\theta};$$

therefore

$$\tfrac{1}{2}h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2 + u^4\,\{\phi'(u)\}^2\left(\frac{du}{d\theta}\right)^2\right] = g\,\{k - \phi(u)\},$$

and differentiating with respect to θ, and dividing by $\frac{du}{d\theta}$,

$$\frac{d^2u}{d\theta^2} + u = -\frac{g}{h^2}\,\phi'(u) - u^2\,\phi'(u)\,\frac{d}{d\theta}\left\{u^2\,\phi'(u)\,\frac{du}{d\theta}\right\},$$

the differential equation of the projection of the path on a horizontal plane.

If we omit the term containing g, we see that the above equation will represent the projection of a geodesic on the given surface.

190. *Suppose the motion to take place in a spherical bowl; or let the particle be suspended by a string from a fixed point.*

This is the most general motion of the *Simple Pendulum.*

Let us take the centre as origin, and the axis of z vertically downwards.

Then $\qquad F(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$

is the equation of constraint, and the equations of motion are

$$\frac{d^2x}{dt^2} = -R\frac{x}{a},$$

$$\frac{d^2y}{dt^2} = -R\frac{y}{a},$$

$$\frac{d^2z}{dt^2} = g - R\frac{z}{a}.$$

Hence, $\frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}=g\,(z-z_0)$ ....(1),

if the motion be due to the level z_0.

But $$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=0\,;$$

or, $$x\frac{dy}{dt}-y\frac{dx}{dt}=h\text{..........................}(2).$$

Also $$x\frac{dx}{dt}+y\frac{dy}{dt}=-z\frac{dz}{dt}\text{ ....}(3),$$

by the equation of the surface.

Squaring and adding (2) and (3),

$$(x^2+y^2)\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}=h^2+z^2\left(\frac{dz}{dt}\right)^2,$$

or $$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=\frac{h^2+a^2\left(\frac{dz}{dt}\right)^2}{a^2-z^2}\,;$$

and therefore from (1),

$$a^2\left(\frac{dz}{dt}\right)^2=2g\,(z-z_0)\,(a^2-z^2)-h^2\text{...............}(4),$$

and therefore z is an elliptic function of t.

The motion will be comprised between two horizontal circles, and if the depth of these circles below the centre be $b+c$ and $b-c$, the cubic in z in the right-hand side of (4) must have roots $b+c$ and $b-c$, and if d be the third root

$$\left(\frac{dz}{dt}\right)^2=-\frac{2g}{a^2}\,(z-b-c)\,(z-b+c)\,(z-d).$$

If we suppose the particle initially on the lower circle and put

$$z=(b+c)\cos^2\phi+(b-c)\sin^2\phi$$
$$=b+c\cos 2\phi,$$

then $$z-b-c=-2c\sin^2\phi,$$
$$z-b+c=\quad 2c\cos^2\phi,$$
$$\frac{dz}{dt}=-4c\sin\phi\cos\phi\frac{d\phi}{dt};$$

and therefore

$$\left(\frac{d\phi}{dt}\right)^2=\frac{g}{2a^2}(z-d)$$

$$=\frac{g}{2a^2}(b+c-d-2c\sin^2\phi)$$

$$=\frac{gc}{a^2k^2}(1-k^2\sin^2\phi),$$

where $$k^2=\frac{2c}{b+c-d};$$

and therefore $$\phi=\operatorname{am}\left(K\frac{t}{T},k\right),$$

where $$\frac{K^2}{T^2}=\frac{gc}{a^2k^2}=\frac{1}{2}\frac{g}{a^2}(b+c-d).$$

Therefore the vertical motion of the bob of the pendulum will be the same as that of a point on a simple circular pendulum of length $\frac{a^2}{c}$ performing complete revolutions in the same periodic time $2T$ as the spherical pendulum.

Now $$(z-b-c)(z-b+c)(z-d)$$
$$=(z-z_0)(z^2-a^2)+\frac{h^2}{2g}$$
$$=z^3-z_0z^2-a^2z+a^2z_0+\frac{h^2}{2g};$$

therefore $$2b+d=z_0,$$
$$b^2-c^2+2bd=-a^2,$$
$$(b^2-c^2)d=-a^2z_0-\frac{h^2}{2g}.$$

Therefore $$d = -\frac{a^2+b^2-c^2}{2b},$$

and $$k^2 = \frac{4bc}{2b(b+c)+a^2+b^2-c^2}$$

$$= \frac{4bc}{a^2+(b+c)(3b-c)}.$$

If ψ denote the angle which the vertical plane through the pendulum has turned through in the time t, then

$$x\frac{dy}{dt} - y\frac{dx}{dt} = (x^2+y^2)\frac{d\psi}{dt} = h,$$

or $$\frac{d\psi}{dt} = \frac{h}{a^2-z^2}$$

$$= \frac{1}{2}\frac{h}{a}\left(\frac{1}{a+z} + \frac{1}{a-z}\right).$$

Now $$z = b + c - 2c\sin^2\phi$$

$$= b + c - 2c\,\mathrm{sn}^2 u,$$

putting $u = \dfrac{K}{T}t$; and therefore

$$\frac{d\psi}{du} = \frac{1}{2}\frac{T}{K}\frac{h}{a}\left(\frac{\dfrac{1}{a+b+c}}{1-\dfrac{2c}{a+b+c}\,\mathrm{sn}^2 u} + \frac{\dfrac{1}{a-b-c}}{1+\dfrac{2c}{a-b-c}\,\mathrm{sn}^2 u}\right).$$

To reduce these expressions to Jacobi's normal form of the third elliptic integral, we must put

$$K^2\,\mathrm{sn}^2(i\alpha_1 + K) = \frac{2c}{a+b+c}, \quad K^2\,\mathrm{sn}^2\, i\alpha_2 = -\frac{2c}{a-b-c};$$

and then α_1 and α_2 will be real.

Therefore $$\mathrm{sn}^2(i\alpha_1 + K) = \frac{b+c-d}{a+b+c},$$

$$\operatorname{cn}^2(ia_1+K)=\frac{a+d}{a+b+c},$$

$$\operatorname{dn}^2(ia_1+K)=\frac{a+b-c}{a+b+c};$$

$$\operatorname{sn}^2 ia_2=-\frac{b+c-d}{a-b-c},$$

$$\operatorname{cn}^2 ia_2=\frac{a-d}{a-b-c},$$

$$\operatorname{dn}^2 ia_2=\frac{a-b+c}{a-b-c}.$$

Now
$$\frac{h^2}{2g}=-(b^2-c^2)\,d-a^2z_0$$
$$=-(b^2-c^2)\,d+(b^2-c^2+2bd)\,(2b+d)$$
$$=2b\,(b+c+d)\,(b-c+d),$$

or
$$\frac{h^2}{a^2}=4\frac{bg}{a^2}(b+c+d)\,(b-c+d)$$
$$=8\frac{K^2}{T^2}b\frac{(b+c+d)\,(b-c+d)}{b+c-d}$$
$$=4\frac{K^2}{T^2}\frac{(a+b+c)\,(a+b-c)\,(a-b+c)\,(a-b-c)}{b\,(b+c-d)};$$

and therefore
$$\frac{1}{4}\frac{h^2}{a^2}\frac{1}{(a+b+c)^2}=\frac{K^2}{T^2}\frac{(a+b-c)\,(a-b+c)\,(a-b-c)}{b\,(b+c-d)\,(a+b+c)}$$
$$=-\frac{K^2}{T^2}\frac{\operatorname{cn}^2(ia_1+K)\operatorname{dn}^2(ia_1+K)}{\operatorname{sn}^2(ia_1+K)};$$

and similarly,
$$\frac{1}{4}\frac{h^2}{a^2}\frac{1}{(a-b-c)^2}=-\frac{K^2}{T^2}\frac{\operatorname{cn}^2 ia_2\operatorname{dn}^2 ia_2}{\operatorname{sn}^2 ia_2}.$$

Therefore

$$\frac{d\psi}{du} = i\frac{\dfrac{\operatorname{cn}(ia_1+K)\operatorname{dn}(ia_1+K)}{\operatorname{sn}(ia_1+K)}}{1-K^2\operatorname{sn}^2(ia_1+K)\operatorname{sn}^2 u} + i\frac{\dfrac{\operatorname{cn} ia_2 \operatorname{dn} ia_2}{\operatorname{sn} ia_2}}{1-K^2\operatorname{sn}^2 ia_2 \operatorname{sn}^2 u},$$

and therefore, measuring ψ from the lowest position of the pendulum,

$$\psi = i\left\{\frac{\operatorname{cn}(ia_1+K)\operatorname{dn}(ia_1+K)}{\operatorname{sn}(ia_1+K)} + \frac{\operatorname{cn} ia_2 \operatorname{dn} ia_2}{\operatorname{sn} ia_2}\right\}u$$

$$+ i\Pi(u, ia_1+K) + i\Pi(u, ia_2)$$

$$= \left\{\frac{d\log\operatorname{sn}(ia_1+K)}{da_1} + \frac{d\log\operatorname{sn} ia_2}{da_2}\right.$$

$$\left.+\frac{d\log\Theta(ia_1+K)}{da_1} + \frac{d\log\Theta\, ia_2}{da_2}\right\}u$$

$$+\frac{1}{2}i\log\frac{\Theta(u-ia_1-K)\,\Theta(u-ia_2)}{\Theta(u+ia_1+K)\,\Theta(u+ia_2)}$$

$$= \left\{\frac{d\log H(ia_1+K)}{da_1} + \frac{d\log H\, ia_2}{da_2}\right\}u$$

$$+\frac{1}{2}i\log\frac{\Theta(u-ia_1-K)\,\Theta(u-ia_2)}{\Theta(u+ia_1+K)\,\Theta(u+ia_2)}$$

$$= \left\{\pi\frac{a_1+a_2}{2KK'} + \frac{d\log\Theta(a_1, k')}{da_1} + \frac{d\log H(a_2, k')}{da_2}\right\}u$$

$$+\frac{1}{2}i\log\frac{\Theta(u-ia_1-K)\,\Theta(u-ia_2)}{\Theta(u+ia_1+K)\,\Theta(u+ia_2)},$$

where $u = K\dfrac{t}{T}$, and k' is the modulus complementary to k, $=\sqrt{1-k^2}$.

If we put $$\psi = \Psi \frac{t}{T} + \psi',$$

where $$\frac{\Psi}{K} = \pi \frac{a_1 + a_2}{2KK'} + \frac{d \log \Theta (a_1, k')}{da_1} + \frac{d \log H (a_2, k')}{da_2},$$

and $$\psi' = \frac{1}{2} i \log \frac{\Theta (u - ia_1 - K) \Theta (u - ia_2)}{\Theta (u + ia_1 + K) \Theta (u + ia_2)},$$

then Ψ will be the apsidal angle, and $\frac{\Psi}{T}$ the mean angular velocity of the vertical plane through the pendulum: also ψ', the periodic part of ψ, will be such that

$$e^{2i\psi'} = \frac{\Theta (u + ia_1 + K) \Theta (u + ia_2)}{\Theta (u - ia_1 - K) \Theta (u - ia_2)};$$

and therefore

$$\cos 2\psi' = \frac{1}{2} \frac{\Theta (u+ia_1+K) \Theta (u+ia_2)}{\Theta (u-ia_1-K) \Theta (u-ia_2)} + \frac{1}{2} \frac{\Theta (u-ia_1-K) \Theta (u-ia_2)}{\Theta (u+ia_1+K) \Theta (u+ia_2)},$$

a periodic function of t, of period T.

191. An interesting special case is that of the *Conical Pendulum*, as it is called, when the particle moves in a horizontal plane and therefore in a circular path, the string describing a right circular cone whose axis is vertical.

Here z is constant and equal to b, c being zero; and

therefore $$\frac{d\psi}{dt} = \frac{h}{a^2 - b^2},$$

where $$\frac{h^2}{2g} = \frac{(a^2 - b^2)^2}{2b};$$

and therefore $$\frac{d\psi}{dt} = \sqrt{\frac{g}{b}},$$

and the time of a complete revolution is

$$2\pi \sqrt{\frac{b}{g}},$$

depending only on the depth of the plane of motion below the point of suspension.

If the motion be slightly disturbed the period of a vibration, putting

$$k=0, \quad K=\frac{1}{2}\pi, \quad \text{and}\ \frac{k^2}{c}=\frac{4b}{a^2+3b^2},$$

becomes

$$\frac{2\pi a}{\sqrt{g}} \cdot \frac{\sqrt{b}}{\sqrt{a^2+3b^2}},$$

and the apsidal angle of the projection of the motion on a horizontal plane

$$2\pi\frac{a}{\sqrt{a^2+3b^2}}.$$

192. *To determine the nature of the small oscillations executed by a particle, under gravity, about a position of stable equilibrium on a smooth surface.*

The tangent plane at the position of equilibrium must be horizontal, and the contiguous portion of the surface must be synclastic and evidently lie above the tangent plane in order that the equilibrium may be stable.

If ρ, ρ_1 be the radii of curvature of the principal normal sections, and if the axes of x and y be tangents to these sections respectively, at the point of contact with the horizontal plane, we know by Analytical Geometry that the equation of the surface in the immediate neighbourhood of the origin is of the form

$$2z-\frac{x^2}{\rho}-\frac{y^2}{\rho_1}=0 \quad \text{.......................(1).}$$

The equations of motion of the particle are, as in § 187,

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=R\lambda\\ \frac{d^2y}{dt^2}&=R\mu\\ \frac{d^2z}{dt^2}&=R\nu-g\end{aligned}\right\} \quad \text{.............................(2).}$$

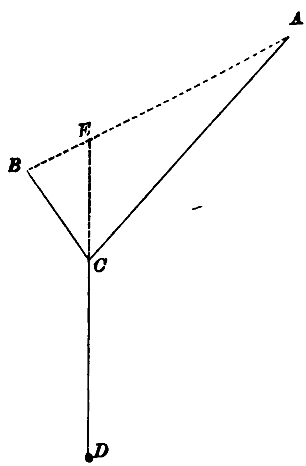
If x and y are small, z is of the second order of small quantities by (1) and may therefore be neglected, as may also $\frac{d^2z}{dt^2}$.

Hence $\lambda = -\frac{x}{\rho}$, $\mu = -\frac{y}{\rho_1}$, $\nu = 1$, approximately. Eliminating R from equations (2), we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{g}{\rho}x\\ \frac{d^2y}{dt^2} &= -\frac{g}{\rho_1}y\end{aligned}\right\}\dots\dots\dots\dots(3),$$

which show (§ 177) that the motion consists of simultaneous simple pendulum small oscillations in the principal planes, the lengths of the pendulums being the corresponding radii of curvature.

The annexed cut shows a very simple arrangement, due to Prof. Blackburn of Glasgow, by which this species of con-



straint may easily be produced. Three strings are knotted together at the point C, the other ends A and B of two of

them are attached to fixed points, and the third supports the particle D. Suppose CE to be vertical, then the small oscillations of D will evidently be executed as if on a smooth surface whose principal planes of curvature at D are in, and perpendicular to, the plane of the paper. The radii of curvature in these planes are CD and DE respectively.

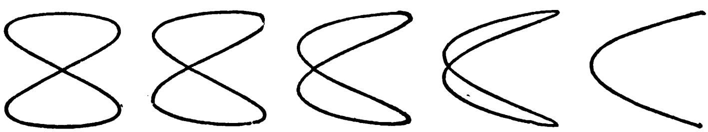
If we put $\frac{g}{\rho}=n^2$, and $\frac{g}{\rho_1}=n_1^2$, the integrals of (3) are

$$\left.\begin{aligned} x &= A\cos(nt+B) \\ y &= A_1\cos(n_1t+B_1) \end{aligned}\right\} \text{.........................} (4).$$

The curves corresponding to these equations are very interesting, but we cannot enter at length on the consideration of them. We may take, as a special case, that in which $DE=4CD$; in which therefore

$$\left.\begin{aligned} x &= A\cos(nt+B) \\ y &= A_1\cos(2nt+B_1) \end{aligned}\right\} \text{.........................} (5).$$

The circumstances of projection determine in each case the particular curve described—a few of the principal forms are sketched below, the last of which is a portion of a parabola.



When n_1 is nearly, but not exactly, equal to $2n$, the curve described is always for a short time approximately one of the above figures, but its form slowly passes in succession from one member of the series to the next, completing the round when one pendulum has executed one more or less than twice as many complete oscillations as the other.

193. We must next consider the effect of the earth's rotation upon the motion of a simple pendulum. Strange to say it was left for Foucault to point out, in February 1851, that the plane of vibration of a simple pendulum suspended at either pole would *appear* to turn through 4

right angles in 24 hours—the plane, in fact, remaining constant in position while objects beneath the pendulum were carried round by the diurnal rotation. At the equator, it was pretty obvious that no such effect would occur, at least if the original plane of vibration was east and west. By some process, of which he gives no account, he arrived at the result that the plane of oscillation must, in any latitude, appear to make a complete revolution in $24^h \times$ cosec. lat. This curious result has been amply verified by experiment.

194. The equations of motion of the pendulum, referred to rectangular axes fixed in space and drawn from the earth's centre, the polar axis being that of z, are obviously

$$m\frac{d^2x}{dt^2} = -T\frac{x-a}{l} + mX,$$

with similar expressions in y and z; a, b, c, being the co-ordinates of the point of suspension, T the tension, l the length of the string, and X, Y, Z the components of gravity.

The equations of motion referred to a new set of axes, parallel to the former, but drawn through the point of suspension, are

$$\left.\begin{aligned} m\frac{d^2(x-a)}{dt^2} &= -T\frac{x-a}{l} + m\left(X - \frac{d^2a}{dt^2}\right) \\ \&\text{c.} &= \quad \&\text{c.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (1).$$

Let us now refer the motion to axes turning with the earth, but drawn from the point of suspension. If the axis of ξ be drawn vertically, and the axes of η, ζ respectively southwards and eastwards; and if ωt be the angle at time t between the planes of xz and $\xi\eta$, λ being the co-latitude of the point of suspension, we have at once (assuming that ξ intersects z)

$$\cos \widehat{x\xi} = \sin\lambda\cos\omega t, \quad \cos \widehat{x\eta} = \cos\lambda\cos\omega t, \quad \cos \widehat{x\zeta} = -\sin\omega t,$$

$$\cos \widehat{y\xi} = \sin\lambda\sin\omega t, \quad \cos \widehat{y\eta} = \cos\lambda\sin\omega t, \quad \cos \widehat{y\zeta} = \cos\omega t,$$

$$\cos \widehat{z\xi} = \cos\lambda, \quad \cos \widehat{z\eta} = -\sin\lambda, \quad \cos \widehat{z\zeta} = 0.$$

By means of these expressions we can at once find the values of $x-a$, $y-b$, $z-c$ in terms of ξ, η, ζ, t, as follows:

$$x-a=\xi\sin\lambda\cos\omega t+\eta\cos\lambda\cos\omega t-\zeta\sin\omega t,$$
$$y-b=\xi\sin\lambda\sin\omega t+\eta\cos\lambda\sin\omega t+\zeta\cos\omega t,$$
$$z-c=\xi\cos\lambda\qquad-\eta\sin\lambda.$$

Let γ be the acceleration due to the attraction of gravity alone, and ν the angle (nearly equal to λ) which its direction makes with the polar axis. [We have above in effect assumed that its direction lies in the plane of $z\xi$, as we have assumed that the axis of ξ intersects the polar axis, while we know that the centrifugal force lies in their common plane.] Let r be the distance of the point of suspension from the earth's centre, μ the angle its direction makes with the polar axis. Then

$$a=r\sin\mu\cos\omega t,\quad b=r\sin\mu\sin\omega t,\quad c=r\cos\mu.$$

With these data equations (1) become

$$\begin{aligned}&\sin\lambda\left[\left(\frac{d^2\xi}{dt^2}-\xi\omega^2\right)\cos\omega t-2\omega\frac{d\xi}{dt}\sin\omega t\right]\\&+\cos\lambda\left[\left(\frac{d^2\eta}{dt^2}-\eta\omega^2\right)\cos\omega t-2\omega\frac{d\eta}{dt}\sin\omega t\right]\\&-\left(\frac{d^2\zeta}{dt^2}-\zeta\omega^2\right)\sin\omega t-2\omega\frac{d\zeta}{dt}\cos\omega t\\&\qquad=-\frac{T}{lm}(\xi\sin\lambda\cos\omega t+\eta\cos\lambda\cos\omega t-\zeta\sin\omega t)\\&\qquad\qquad-\gamma\sin\nu\cos\omega t+r\omega^2\sin\mu\cos\omega t.\end{aligned}$$

$$\begin{aligned}&\sin\lambda\left[\left(\frac{d^2\xi}{dt^2}-\xi\omega^2\right)\sin\omega t+2\omega\frac{d\xi}{dt}\cos\omega t\right]\\&+\cos\lambda\left[\left(\frac{d^2\eta}{dt^2}-\eta\omega^2\right)\sin\omega t+2\omega\frac{d\eta}{dt}\cos\omega t\right]\\&+\left(\frac{d^2\zeta}{dt^2}-\zeta\omega^2\right)\cos\omega t-2\omega\frac{d\zeta}{dt}\sin\omega t\\&\qquad=-\frac{T}{lm}(\xi\sin\lambda\sin\omega t+\eta\cos\lambda\sin\omega t+\zeta\cos\omega t)\\&\qquad\qquad-\gamma\sin\nu\sin\omega t+r\omega^2\sin\mu\sin\omega t.\end{aligned}$$

$$\frac{d^2\xi}{dt^2}\cos\lambda - \frac{d^2\eta}{dt^2}\sin\lambda = -\frac{T}{lm}(\xi\cos\lambda - \eta\sin\lambda) - \gamma\cos\nu.$$

As we contemplate small vibrations only, we may treat ξ as being practically equal to $-l$, and omit its differential coefficients. We also omit powers and products of η, ζ, and terms in ω^2, except those in which it is multiplied by a large quantity. For it is known that the centrifugal force at the equator is about $\frac{1}{289}$th of gravity, or that approximately

$$r\omega^2 = \frac{g}{289}.$$

With these simplifications our equations become

$$\cos\lambda\left(\frac{d^2\eta}{dt^2}\cos\omega t - 2\omega\frac{d\eta}{dt}\sin\omega t\right) - \frac{d^2\zeta}{dt^2}\sin\omega t - 2\omega\frac{d\zeta}{dt}\cos\omega t$$

$$= -\frac{T}{lm}(-l\sin\lambda\cos\omega t + \eta\cos\lambda\cos\omega t - \zeta\sin\omega t)$$

$$-\gamma\sin\nu\cos\omega t + r\omega^2\sin\mu\cos\omega t.$$

$$\cos\lambda\left(\frac{d^2\eta}{dt^2}\sin\omega t + 2\omega\frac{d\eta}{dt}\cos\omega t\right) + \frac{d^2\zeta}{dt^2}\cos\omega t - 2\omega\frac{d\zeta}{dt}\sin\omega t$$

$$= -\frac{T}{lm}(-l\sin\lambda\sin\omega t + \eta\cos\lambda\sin\omega t + \zeta\cos\omega t)$$

$$-\gamma\sin\nu\sin\omega t + r\omega^2\sin\mu\sin\omega t.$$

$$-\frac{d^2\eta}{dt^2}\sin\lambda = \frac{T}{lm}(l\cos\lambda + \eta\sin\lambda) - \gamma\cos\nu.$$

The two first may be put in the form

$$\frac{d^2\eta}{dt^2}\cos\lambda - 2\omega\frac{d\zeta}{dt} = -\frac{T}{lm}(-l\sin\lambda + \eta\cos\lambda) - \gamma\sin\nu + r\omega^2\sin\mu.$$

$$-2\omega\frac{d\eta}{dt}\cos\lambda - \frac{d^2\zeta}{dt^2} = \frac{T}{lm}\zeta.$$

But, when $\eta = 0$, $\zeta = 0$, we have $T = mg$, so that

$$g\sin\lambda - \gamma\sin\nu + r\omega^2\sin\mu = 0,$$

$$g\cos\lambda - \gamma\cos\nu = 0,$$

and our equations become

$$\frac{d^2\eta}{dt^2}\cos\lambda - 2\omega\frac{d\zeta}{dt} = \left(\frac{T}{m} - g\right)\sin\lambda - \frac{T}{lm}\eta\cos\lambda$$

$$-2\omega\frac{d\eta}{dt}\cos\lambda - \frac{d^2\zeta}{dt^2} = \frac{T}{lm}\zeta$$

$$-\frac{d^2\eta}{dt^2}\sin\lambda = \left(\frac{T}{m} - g\right)\cos\lambda + \frac{T}{lm}\eta\sin\lambda.$$

The first and last give

$$\frac{T}{m} - g = -2\omega\frac{d\zeta}{dt}\sin\lambda,$$

and therefore, to the degree of approximation above determined upon,

$$\left.\begin{aligned}\frac{d^2\eta}{dt^2} - 2\omega\cos\lambda\frac{d\zeta}{dt} + \frac{g}{l}\eta = 0\\ \frac{d^2\zeta}{dt^2} + 2\omega\cos\lambda\frac{d\eta}{dt} + \frac{g}{l}\zeta = 0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(2).$$

These are the equations of the motion of the bob, referred to a horizontal plane fixed to the earth. If we omit the middle terms, which obviously depend upon the earth's rotation, we fall back upon the equations of § 192.

195. To interpret equations (2) it is convenient to employ a second change of co-ordinates—to refer the motion to axes revolving uniformly in the plane of η, ζ, with angular velocity Ω. If $\mathfrak{y}, \mathfrak{z}$ be the co-ordinates referred to the new axes, we have by Analytical Geometry

$$\eta = \mathfrak{y}\cos\Omega t - \mathfrak{z}\sin\Omega t, \quad \zeta = \mathfrak{y}\sin\Omega t + \mathfrak{z}\cos\Omega t,$$

the substitution of which in (2) leads to the equations

$$\frac{d^2\mathfrak{y}}{dt^2} + \frac{g}{l}\mathfrak{y} = 0, \qquad \frac{d^2\mathfrak{z}}{dt^2} + \frac{g}{l}\mathfrak{z} = 0\ldots\ldots\ldots\ldots\ldots(3),$$

if we make the assumption

$$\Omega = -\omega\cos\lambda\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and omit as before terms of the order ω^2.

(4) shews that the new axes rotate, in the *opposite* direction to that of the earth, with the component of the earth's angular velocity about the vertical at the place. And in the plane, so revolving, we see by (3) that the bob of the pendulum describes an approximately elliptic orbit.

A *circular* path being obviously possible, let us assume as particular integrals of (2)

$$\eta = a\cos(pt+\alpha), \qquad \zeta = a\sin(pt+\alpha).$$

The substitution of these values gives the same result

$$p^2 + 2\omega p\cos\lambda - \frac{g}{l} = 0$$

in each of equations (2).

Put $\frac{g}{l} = n^2$, then the values of p are $n \pm \omega\cos\lambda$, so that the (apparent) angular velocity of a conical pendulum is increased or diminished by $\omega\cos\lambda$ according as its direction of rotation is negative or positive.

196. *A particle under any forces, and resting on a smooth horizontal plane, is attached by an inextensible string to a point which moves in a given manner in that plane; to determine the motion of the particle.*

Let x, y, $\bar{x}$, $\bar{y}$ be the co-ordinates, at time t, of the particle and point, a the length of the string, R the tension of the string, and m the mass of the particle.

For the motion of the particle we have

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= mX - R\frac{x-\bar{x}}{a} \\ m\frac{d^2y}{dt^2} &= mY - R\frac{y-\bar{y}}{a} \end{aligned}\right\} \qquad (1),$$

with the condition $(x-\bar{x})^2 + (y-\bar{y})^2 = a^2$.

Now $\bar{x}$, $\bar{y}$ are given functions of t. Take from both sides of the equations in (1) the quantities $\frac{d^2\bar{x}}{dt^2}$, $\frac{d^2\bar{y}}{dt^2}$, re-

spectively, and we have the equations of *relative* motion

$$\left.\begin{aligned} m\frac{d^2(x-\bar{x})}{dt^2} &= mX - R\frac{x-\bar{x}}{a} - m\frac{d^2\bar{x}}{dt^2} \\ m\frac{d^2(y-\bar{y})}{dt^2} &= mY - R\frac{y-\bar{y}}{a} - m\frac{d^2\bar{y}}{dt^2} \end{aligned}\right\} \dots\dots (2).$$

These are precisely the equations we should have had if the point had been fixed, and in addition to the forces X, Y and R acting on the particle, we had applied, reversed in direction, the accelerations of the point's motion with the mass as a factor. It is evident that the same theorem will hold in three dimensions. The accelerations $\frac{d^2\bar{x}}{dt^2}$, $\frac{d^2\bar{y}}{dt^2}$ are known as functions of t, and therefore the equations of relative motion are completely determined.

197. *Let there be no impressed forces, and suppose first that the point moves with constant velocity in a straight line.*

Here $\frac{d\bar{x}}{dt}$, $\frac{d\bar{y}}{dt}$ are constant, and therefore no terms are introduced in the equations of motión. We have thus the case of § 28.

Again, *suppose the point's motion to be rectilinear, but uniformly accelerated.*

The relative motion will evidently be that of a simple pendulum from side to side of the point's line of motion. In certain cases, when the angular velocity exceeds a certain limit, we shall have the string occasionally untended; and this will give rise to an impact when it is again tended. While the string is untended the particle moves, of course, in a straight line.

198. *Suppose the point to move, with constant angular velocity ω, in a circle whose radius is* r *and centre origin.*

Here, supposing the point to start from the axis of x,

$$\bar{x} = r\cos\omega t,\quad \bar{y} = r\sin\omega t.$$

Hence the equations of motion are, since

$$\frac{d^2\bar{x}}{dt^2} = -\omega^2\bar{x}, \quad \frac{d^2\bar{y}}{dt^2} = -\omega^2\bar{y},$$

$$\left.\begin{aligned} \frac{d^2(x-\bar{x})}{dt^2} &= -R\frac{x-\bar{x}}{a} + \omega^2\bar{x}, \\ \frac{d^2(y-\bar{y})}{dt^2} &= -R\frac{y-\bar{y}}{a} + \omega^2\bar{y}, \\ (x-\bar{x})^2 + (y-\bar{y})^2 &= a^2. \end{aligned}\right\}$$

Whence $(x-\bar{x})\dfrac{d^2(y-\bar{y})}{dt^2} - (y-\bar{y})\dfrac{d^2(x-\bar{x})}{dt^2}$

$$= \omega^2\{(x-\bar{x})\,\bar{y} - (y-\bar{y})\,\bar{x}\};$$

or, in polar co-ordinates, for the relative motion,

$$\frac{d}{dt}\left(a^2\frac{d\theta}{dt}\right) = -\omega^2 ar\sin(\theta-\omega t),$$

or
$$\frac{d^2(\theta-\omega t)}{dt^2} = -\omega^2\frac{r}{a}\sin(\theta-\omega t).$$

Now $\theta - \omega t$ is the inclination of the string to the radius passing through the point; call it ϕ, and we have

$$\frac{d^2\phi}{dt^2} = -\omega^2\frac{r}{a}\sin\phi,$$

which is the ordinary equation of motion of a simple pendulum whose length is $\dfrac{ga}{r\omega^2}$.

The particle therefore moves, with reference to the uniformly revolving radius of the circle described by the point, just as a simple pendulum with reference to the vertical.

199. *To determine the motion of a particle under given forces, moving in a smooth tube, in the form of a given plane curve, which revolves in a given manner about an axis in its plane.*

Let the axis of revolution be that of z, and let the position of the particle at time t be given by its distance r from that

axis, the plane of the tube at that instant making an angle θ with a fixed plane passing through the axis. By the conditions of the problem θ is a given function of t.

The sole effect of the tube will be to produce a pressure of constraint, which lies in the normal plane to the tube, and may therefore be resolved into two parts, one perpendicular to the plane of the tube, the other in that plane and in the principal normal to the tube.

Let the impressed forces be resolved into three, P along r, T perpendicular to the plane of the tube, and S parallel to the axis of z.

Let R, R' be the two resolved parts of the pressure of constraint.

The equations of motion will then be (by §§ 16, 69)

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = P + R\frac{dz}{ds} \quad\text{.................. (1)},$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = T - R' \quad\text{...................... (2)},$$

$$\frac{d^2z}{dt^2} = S - R\frac{dr}{ds} \quad\text{.......................... (3)},$$

where s is the arc of the revolving curve.

In addition to these we have the two equations

$$\theta = f(t) \quad\text{............................ (4)},$$

which gives the position of the tube at any time, and

$$r = \phi(z) \quad\text{............................ (5)},$$

the equation of the tube.

By means of (4) and (5) we may eliminate θ, r, and s from (1), (2), (3). Then eliminating R between (1) and (3), we obtain a differential equation between z and t, whose integral together with (4) completely determines the *position* of the particle at any instant.

R and R' may then be found from (1) or (3), and (2).

In the simplest case when the angular velocity of the tube is constant, or $\frac{d\theta}{dt}=\omega$, (4) becomes $\theta=\omega t$ if the plane from which θ is measured be that of the tube at the time $t=0$.

We proceed to give an example or two.

200. *A particle moves in a smooth straight tube which revolves with constant angular velocity round a vertical axis to which it is perpendicular, to determine the motion.*

Here $z=$ constant, $\frac{d\theta}{dt}=$ constant $=\omega$, $P=0$, and we have from (1)

$$\frac{d^2r}{dt^2}-r\omega^2=0;$$

whence
$$r=A\epsilon^{\omega t}+B\epsilon^{-\omega t}.$$

Suppose the motion to commence at time $t=0$ by the cutting of a string, length a, attaching the particle to the axis. The velocity of the particle at that instant along the tube would be zero. Hence at $t=0$

$$r=a=A+B,$$

$$\frac{dr}{dt}=0=A-B;$$

$$\therefore\ A=B=\frac{1}{2}a;$$

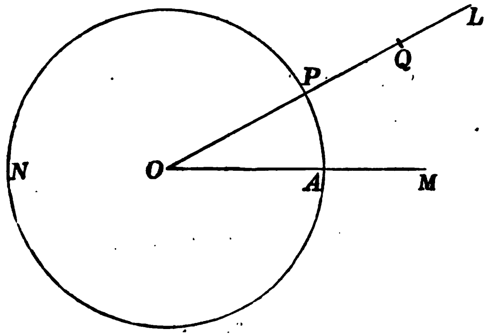
and
$$r=\frac{1}{2}a\left(\epsilon^{\omega t}+\epsilon^{-\omega t}\right).$$

In the figure, let OM be the initial position of the tube, A that of the particle; OL, Q the tube and particle at time t. Then $OA=a$, arc $AP=a\omega t$, $OQ=r$, and we have

$$OQ=\frac{1}{2}OA\left(\epsilon^{\frac{\text{arc }AP}{OA}}+\epsilon^{-\frac{\text{arc }AP}{OA}}\right).$$

From this we see that OQ and the arc AP are corresponding values of the ordinate and abscissa of a catenary whose

parameter is OA. (It is not necessary for the tube to meet the axis of revolution.)



Here, by (3), we have evidently $R = g$.

Also, by (2), $R' = -2\frac{a\omega}{2}(\epsilon^{\omega t} - \epsilon^{-\omega t})\,\omega$

$$= -\omega^2 a\,(\epsilon^{\omega t} - \epsilon^{-\omega t}).$$

From this equation, combined with the value of r, we easily deduce

$$R' = 2\omega^2\sqrt{(r^2 - a^2)},$$

and it is therefore proportional at any instant to the tangent drawn from Q to the circle APN.

201. *Suppose the tube to revolve with constant angular velocity in a vertical plane about a horizontal axis.*

We have from equation (1) of § 199

$$\frac{d^2r}{dt^2} - r\omega^2 = -g\cos\omega t,$$

if we conceive the tube to be vertical when $t = 0$. The integral of this equation is

$$r = A\epsilon^{\omega t} + B\epsilon^{-\omega t} - g\left\{\left(\frac{d}{dt}\right)^2 - \omega^2\right\}^{-1}\cos\omega t,$$

or $$r = A\epsilon^{\omega t} + B\epsilon^{-\omega t} + \frac{g}{2\omega^2}\cos\omega t;$$

and if $$r = a,\ \frac{dr}{dt} = 0,\ \text{when } t = 0,$$

we have $$a = A + B + \frac{g}{2\omega^2},$$

and $$0 = A - B;$$

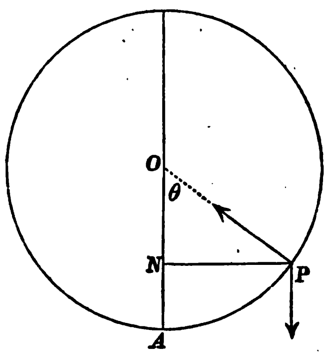
or, $$r = \left(\frac{a}{2} - \frac{g}{4\omega^2}\right)(\epsilon^{\omega t} + \epsilon^{-\omega t}) + \frac{g}{2\omega^2}\cos\omega t,$$

which completely determines the motion. R and R' may be found as before.

202. *Let the tube be in the form of a circle turning with constant angular velocity about a vertical diameter.*

Let AO be the axis, P the position of the particle at any time. Let $POA = \theta$ denote the particle's position. The accelerations of the particle in the directions ON and NP being

$$\frac{d^2 ON}{dt^2} \text{ and } \frac{d^2 NP}{dt^2} - \omega^2 NP;$$



therefore $$a\frac{d^2\cos\theta}{dt^2} = g - R\cos\theta,$$

$$a\frac{d^2 \sin\theta}{dt^2} - \omega^2 a \sin\theta = -R\sin\theta.$$

Eliminating R

$$a\frac{d^2\theta}{dt^2} - a\omega^2 \sin\theta\cos\theta = -g\sin\theta \dots\dots\dots\dots (1).$$

The position of equilibrium will therefore be given by $\sin\theta = 0$, or $\theta = \gamma$, where $\cos\gamma = \frac{g}{a\omega^2}$.

Integrating (1)

$$\left(\frac{d\theta}{dt}\right)^2 = C + 2\omega^2\cos\gamma\cos\theta - \omega^2\cos^2\theta \dots\dots\dots (2).$$

I. Suppose the particle to be making complete revolutions, passing through the lowest point with velocity $a\omega_1$; therefore

$$\left(\frac{d\theta}{dt}\right)^2 = \omega_1^2 - 2\omega^2\cos\gamma(1-\cos\theta) + \omega^2\sin^2\theta$$

$$= \omega^2\left\{(1-\cos\gamma)^2 + \frac{\omega_1^2}{\omega^2} - (\cos\theta - \cos\gamma)^2\right\},$$

and $\frac{d\theta}{dt}$ can never vanish if $\frac{\omega_1^2}{\omega^2} > 4\cos\gamma$, or $\omega_1^2 > \frac{4g}{a}$, that is, if the velocity at the lowest point be greater than that due to the level of the highest point.

To solve the equation, we must put

$$\tan\frac{\theta}{2} = \sqrt{\frac{s-1}{s+1}}\,\text{tn}\,u, \text{ where } k' = \sqrt{\frac{(r+1)(s+1)}{(r-1)(s-1)}},$$

where $$u = \frac{1}{2}\omega t\sqrt{\{(s+1)(1-r)\}},$$

and s, r are the values of $\cos\theta$ that make the right-hand side of the equation vanish, s being > 1, and $r < -1$.

II. If $\omega_1^2 < \frac{4g}{a}$, the particle will oscillate through the lowest point, and if $\frac{d\theta}{dt} = 0$, when $\theta = \alpha$, then

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{a}(\cos\theta - \cos\alpha) - \omega^2(\cos^2\theta - \cos^2\alpha)$$

$$= \omega^2(\cos\theta - \cos\alpha)\left(\frac{2g}{a\omega^2} - \cos\alpha - \cos\theta\right)$$

$$= \omega^2(\cos\theta - \cos\alpha)(2\cos\gamma - \cos\alpha - \cos\theta),$$

and therefore if

$$2\cos\gamma - \cos\alpha > 1,$$

the particle will oscillate through the lowest point.

We must put

$$\tan\frac{\theta}{2} = \tan\frac{\alpha}{2}\operatorname{cn} u, \text{ where } k' = \cot\frac{\alpha}{2}\sqrt{2\cos\gamma - \cos\alpha - 1},$$

and then $\qquad u = \omega t\sqrt{(\cos\gamma - \cos\alpha)}.$

III. If $\qquad 1 > 2\cos\gamma - \cos\alpha > -1,$

then putting

$$2\cos\gamma - \cos\alpha = \cos\beta,$$

$$\left(\frac{d\theta}{dt}\right)^2 = \omega^2(\cos\theta - \cos\alpha)(\cos\beta - \cos\theta),$$

and the particle will oscillate on one side of the vertical diameter.

We must put $\tan\frac{\theta}{2} = \tan\frac{\alpha}{2}\operatorname{dn} u$, or $\tan\frac{\theta}{2} = \tan\frac{\beta}{2}\frac{1}{\operatorname{dn} u}$,

and then $\qquad u = \omega t\sin\frac{\alpha}{2}\cos\frac{\beta}{2}, \quad k' = \dfrac{\tan\frac{\beta}{2}}{\tan\frac{\alpha}{2}}.$

203. *To find the form of the tube in order that the particle projected with given velocity may preserve its velocity unchanged, gravity acting parallel to the axis.*

Resolving tangentially, and taking co-ordinates x, y in the plane of the curve, the axis of revolution being that of y, we have

$$\frac{d^2s}{dt^2} = x\omega^2\frac{dx}{ds} - g\frac{dy}{ds}.$$

Hence, $$\left(\frac{ds}{dt}\right)^2 = x^2\omega^2 - 2gy + C.$$

But $$\frac{ds}{dt} = \text{constant}.$$

Hence, $$x^2 = \frac{2g}{\omega^2}(y + k),$$

the equation of a parabola whose axis is vertical and vertex downwards. This result might easily have been foreseen, as the velocity can only be constant if the acceleration due to the impressed forces along the curve be zero at every point; that is, if the resultant of gravity and the reaction to circular motion (called the centrifugal force) lie in the normal. That this may be the case, we must have Centrifugal force : Gravity :: Ordinate : Sub-normal. But the centrifugal force is proportional to the ordinate, hence the subnormal must be proportional to gravity, i.e. must be constant: a property peculiar to the parabola. This proposition has a singular application in Hydrostatics.

204. *A particle moves on a rough curve, under given forces; to determine the motion.*

If μ be the coefficient of kinetic friction, and

$$R = \sqrt{(R_1^2 + R_2^2)}$$

be the normal reaction, friction will cause a resistance $\mu\sqrt{(R_1^2 + R_2^2)}$ acting in the tangent to the curve in the opposite direction to the particle's motion.

Equation (1) of § 182 will therefore become

$$\frac{d^2s}{dt^2} = T - \mu\sqrt{(R_1^2 + R_2^2)},$$

the other two equations remaining the same.

If from the three we eliminate R_1 and R_2, we may by means of the equations of the curve eliminate x, y and z, and the final result, involving only s and t, suffices to determine the motion completely.

205. Ex. *A particle moves in a rough tube in the form of a plane curve, under no forces; to determine the motion.*

Here $$\frac{d^2s}{dt^2} = -\mu R = -\frac{\mu v^2}{\rho}.$$

Now $$v\frac{dv}{ds} = \frac{d^2s}{dt^2},$$

hence $$v\frac{dv}{ds} = -\mu\frac{v^2}{\rho}:$$

or $$v = a\epsilon^{-\mu\int\frac{ds}{\rho}}.$$

But, if ψ be the angle which the tangent at any point makes with a fixed line,

$$\frac{ds}{d\psi} = \rho.$$

Hence, $v = a\epsilon^{-\mu\psi}$, where a is the velocity when $\psi = 0$.

It may be instructive to compare this result with that for the tension of a string stretched over a rough curve.

If the curve be tortuous, $\frac{ds}{\rho}$ is the angle between two successive tangents. If the surface of which the curve is the cuspidal edge be developed, and if ϕ represent the angle between the tangents corresponding to the initial and final positions of the particle,

$$v = a\epsilon^{-\mu\phi}.$$

206. *A particle under given forces moves on a given rough surface; to determine the motion.*

If R be the normal reaction of the surface, the friction will cause a resistance $\mu' R$, and the equations of motion become

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= X + R\lambda - \mu' R\frac{dx}{ds} \\ \frac{d^2y}{dt^2} &= Y + R\mu - \mu' R\frac{dy}{ds} \\ \frac{d^2z}{dt^2} &= Z + R\nu - \mu' R\frac{dz}{ds} \end{aligned}\right\},$$

from which R must be eliminated. The two resulting equations contain x, y, z and t, and if the latter be eliminated, we have one equation in x, y, z, which, with the equation of the surface, will completely determine the path. In general these equations are utterly intractable.

EXAMPLES.

(1) If a particle, attached by a string to a point, *just* make complete revolutions in a vertical plane, the tension of the string in the two positions when it is vertical is zero, and six times the weight of the particle, respectively.

(2) A pendulum which vibrates *seconds* at a place A gains n beats in 24 hours at a place B; compare gravity at the two places.

(3) Prove that a seconds pendulum when taken to the top of a mountain h miles high will lose $21 \cdot 6h$ beats in a day nearly.

(4) The times of oscillation of a pendulum are observed at the earth's surface, and also at a height h above the surface; from these data find the radius of the earth supposed spherical.

(5) Shew that a simple circular pendulum under a central attraction varying as the distance will move as it does under gravity.

(6) A pendulum oscillates in a small circular arc, and is acted on in addition to gravity by a small horizontal attraction as the attraction of a mountain. Shew how to find this attraction by observing the number of oscillations gained in a given time. Also find the direction in which the attraction must act so as not to alter the time of oscillation.

(7) Prove that a particle moving under gravity on the convex side of a vertical circle will leave the circle at two-thirds of the height above the centre of the line to the level of which the velocity is due.

(8) A particle is suspended from a fixed point by an inextensible string: find the level to which the velocity must be due, so that the particle after the string has ceased to be stretched may pass through the point of suspension.

(9) Two particles are projected from the same point, in the same direction, and with the same velocity, but at different instants, in a smooth circular tube of small bore whose plane is vertical, to shew that the line joining them constantly touches another circle.

Let the tube be called the circle A, and the horizontal line, to the level of which the velocity is due, L. Let m, m' be simultaneous positions of the particles. Suppose that mm' passes into its next position by turning about O, these two lines will intercept two indefinitely small arcs at m and m' which (by a property of the circle) are in the ratio $mO : Om'$.

Let another circle B be described touching mm' in O, and such that L is the radical axis of A and B. Let a be the distance between their centres, $mp, m'p'$ perpendiculars on L. Let mp cut A again in q and B in r, s.

Then by Geometry,

$$mp \cdot qp = rp \cdot sp,$$

and therefore
$$\begin{aligned} mO^2 &= mr \cdot ms = (mp - rp)(mp - sp) \\ &= mp\,(mp + qp - rp - sp) \\ &= 2a \cdot mp = \frac{a}{g}\,(\text{velocity of } m)^2. \end{aligned}$$

Similarly

$$Om'^2 = 2a \cdot m'p' = \frac{a}{g}\,(\text{velocity of } m')^2.$$

Hence the velocities of m and m' are as $mO : Om'$, and therefore by what we have shewn above about elementary arcs at m and m', the proximate position of mm' is also a tangent to B, which proves the proposition.

It is easily seen from this, that if *one* polygon of a given number of sides can be inscribed in one circle and circum-

scribed about another, an indefinite number can be drawn. For this we have only to suppose a number of particles moving in A with velocities due to a fall from L, and then if they form at any time the angular points of a polygon whose sides touch B, they will continue to do so throughout the motion. This however does not belong to our subject.

(10) Two segments of circles are described on the under side of the same horizontal line, the one subtending at its centre double the angle which the other subtends; if a particle under gravity describes the lower arc, any tangent to the upper arc will cut off from the lower a portion which will be described in half the time of a single vibration.

(11) AB is a vertical diameter of a fine circular tube in which move three equal particles P, Q, Q' of perfect elasticity; P starts from A and Q, Q' in opposite directions from B with such velocities that at the first impact all three have equal velocities; prove that throughout the motion the line joining any pair is either horizontal or passes through one of two fixed points, and that the intervals of time between successive impacts are all equal.

(12) Two equal smooth circles are fixed so as to touch the same horizontal plane at their lowest points, their planes being at different inclinations; two small beads are projected at the same instant along the circles from their lowest points, the velocity of each bead being due to the level of the highest point of the other circle above the horizontal plane; prove that during the motion the beads will always be at the same level.

(13) Prove that the time of vibration from rest to rest of a simple circular pendulum of length a oscillating through an angle 2α is equal to the time of complete revolution of the pendulum of length $a \operatorname{cosec}^2 \frac{1}{2}\alpha$, the velocity being due to the level $2a \operatorname{cosec}^4 \frac{1}{2}\alpha$, above the lowest point.

(14) A bead can slide on a smooth circular arc AB and is attracted by it, with intensity $f(r)$; if it be displaced

from its position of equilibrium, the time of oscillation will be

$$\frac{2\pi}{\sqrt{2\cos\frac{\alpha}{2}f(AC)}},$$

where C is the middle point of AB, and α the angle AC subtends at the centre of the circle.

(15) A string passes through a small hole in a smooth horizontal table, and has equal particles attached to its ends, one hanging vertically and the other lying on the table at a distance a from the hole; the latter is projected with a velocity $\sqrt{ga}$ perpendicular to the string; shew that the other particle will remain at rest, and if it be slightly disturbed the time of a small oscillation will be $2\pi\sqrt{\frac{2a}{3g}}$.

(16) A particle, under gravity, is attached to a fixed point by means of an elastic string of natural length $3a$, the modulus of elasticity being six times the weight of the particle; when the string is at its natural length and the particle vertically above the point of attachment, the particle is projected horizontally with a velocity $3\sqrt{\frac{ag}{2}}$; prove that the angular velocity of the string will be constant, and that the particle will describe the limaçon

$$r = a(4 - \cos\theta).$$

(17) From a point upon the surface of a smooth vertical circular hollow cylinder, and inside, a particle is projected in a direction making an angle α with the generating line through the point; find the velocity of projection that the particle may rise to a given height (h) above the point, and the condition that the highest point may be vertically above the point of projection.

Find the condition that after n revolutions the particle may be again at the point of projection.

(18) A particle slides down a catenary, whose plane is vertical and vertex upwards, the velocity at any point being

due to the level of the directrix; prove that the pressure at any point is inversely as the distance of that point from the directrix.

(19) A particle projected with given velocity, moves under gravity on a curve in a vertical plane; find the nature of the curve that the pressure on it may be constant throughout the motion.

If the pressure on the curve is always n times the weight, prove that the vertical distance between the highest and lowest points of the curve is

$$\frac{2na}{(n^2-1)^2},$$

and that the interval between the instants at which the particle is at the same level is

$$\pi\sqrt{\frac{a}{g}}\frac{n}{(n^2-1)^{\frac{3}{2}}},$$

the length of the curve between two such points being

$$\pi a\frac{2n^2+1}{(n^2-1)^{\frac{3}{2}}}.$$

Determine the nature of the evolute of this curve, which is such that the string of a simple pendulum must be wrapped on it in order that the tension may be constant, and prove the relation between the length of the arc and the vertical ordinate from the upper cusp

$$y = ns + \frac{3}{2}l^{\frac{1}{3}}(l-s)^{\frac{2}{3}} - \frac{3}{2}l,$$

where l is the length of the string.

(20) The major axis of an ellipse being vertical, shew that in order that a particle projected along the concave side of the arc may pass through the centre after leaving the curve, the velocity must be due to the level

$$\frac{8a^2+b^2}{6a\sqrt{3}}$$

above the centre, a and b being the semiaxes of the ellipse.

(21) A particle is initially at rest at a point of the equiangular spiral $r = c\epsilon^{-m\theta}$, distant d from the pole. Shew that if the pole be a centre of attraction $= \frac{\mu}{D^2}$, the time of fall to it is

$$\frac{\pi}{2} \frac{d^{\frac{3}{2}}}{\sqrt{(2\mu)}} \sqrt{\left(1 + \frac{1}{m^2}\right)}.$$

Find the pressure on the curve at any instant.

(22) A particle attached by a string to a point moves on a horizontal plane. A small ring passing round the string moves with constant velocity in a straight line from the point. Shew how to find the equation of the actual path, and shew that the path relative to the ring is a reciprocal spiral.

(23) A particle moves in a circular groove radius a under a central attraction $\propto D^{-2}$ situated at a distance b from the centre of the circle. It is projected from the nearest point with velocity V, shew that for a complete revolution

$$V^2 \not< \frac{4\mu b}{a^2 - b^2}.$$

(24) Prove that if a particle move in a smooth tube under given central attractions, the pressure at any point of the tube will vary as

$$\frac{1}{\rho}\left\{C - \Sigma \frac{1}{p} \frac{d}{dp}(p^2 F)\right\},$$

where $\frac{dF}{dr}$ is the acceleration due to the attraction of any one of the centres, and ρ is the radius of curvature; and hence that the pressure at any point of the tube will vary as the curvature whenever the orbit is such as could be described freely under each of the attractions taken separately.

(25) A particle of mass m moves in a smooth circular tube of radius a under an attraction $m\mu$ times the distance to a point inside the circle at a distance c from the centre. If the particle be placed very nearly at its greatest distance

from the centre of attraction, prove that it will pass over the quadrant ending at its least distance in the time

$$\sqrt{\frac{a}{\mu c}}\log(\sqrt{2}+1).$$

(26) Shew that a particle moving under gravity on a smooth helix whose axis is vertical, makes the first revolution from rest in the time

$$\sqrt{\frac{8\pi a}{g\sin 2\alpha}}.$$

(27) A groove is cut on a right cone of height h, making an angle β with the generating lines. Shew that the time of reaching the base, from a vertical height h_1 below the vertex, by a particle sliding in the groove is

$$\sqrt{\frac{2}{g}}\frac{\sqrt{h-h_1}}{\cos\alpha\cos\beta},$$

where α is the semivertical angle.

(28) A particle under a central repulsion varying as the distance moves in a tube of the form of an epicycloid, the pole being at the centre of repulsion. Shew that the oscillations are tautochronous.

For an attraction, the curve is a hypocycloid.

(29) Prove that the tautochrone when the attraction is as the cube root of the distance from and perpendicular to the axis of x is the hypocycloid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

(30) A particle P is attached by strings to two points A and B in the same horizontal plane, and describes a vertical circle. When the particle is at the lowest point, the string AP is cut and the particle proceeds to describe a horizontal circle; find the ratio of the new tension of BP to the old tension.

(31) A smooth ring slides on a circular wire which revolves with constant angular velocity about a vertical diameter. If the ring be attached to the highest point by

a fine elastic string of natural length equal to the radius of the wire, and be slightly displaced from the lowest point, shew that it will just reach the highest point if the modulus of elasticity is four times the weight of the particle.

(32) A ring slides on a smooth wire bent into the form of a plane vertical curve, and is attached by an elastic string to a fixed point in the plane of the curve; if it start initially from a position in which the string is just not stretched, prove that it will descend through a vertical distance which is a third proportional to the natural length of the string and its extension at the lowest position, supposing that the modulus of elasticity is twice the weight of the ring, and the string is stretched throughout the motion.

(33) Three equal particles are attached to a string of length $4a$, one at its middle point and the others half way between it and the extremities, which are attached to two points in a horizontal line at a distance $a(\sqrt{3}+1)$ from each other; find the position of equilibrium, and shew that if the middle particle receive a slight vertical displacement the time of a small oscillation is the same as that of a pendulum of length

$$\frac{3-\sqrt{3}}{3}a.$$

(34) A particle, under gravity, is suspended by a light elastic thread which passes through a ring B above the particle and is attached to a fixed point A, AB being the natural length of the string.

If the particle be projected from any point in any direction, prove that it will describe an ellipse about the position of equilibrium of the particle as centre.

Prove that the same will hold if the particle be suspended in a similar way by a number of elastic strings.

(35) A chord AB of a circle is vertical and the inclination of the tangents at A and B to the horizon is the angle of friction. Shew that the time down any chord AC or CB drawn in the smaller of the two segments into which AB divides the circle is constant.

(36) A particle, under no forces, is projected with velocity V in a rough tube in the form of an equiangular spiral at a distance a from the pole and towards the pole; shew that it will arrive at the pole in time

$$\frac{a}{V}\frac{1}{\cos\alpha - \mu\sin\alpha},$$

α being the angle of the spiral and μ ($< \cot\alpha$) the coefficient of friction.

(37) A bead is projected along a rough plane curved wire, such that it changes the direction of its motion with constant angular velocity. Shew that the form of the wire must be a logarithmic spiral.

(38) A particle attached to a point by a string whose natural length is a, lies on a rough horizontal plane and is projected perpendicular to the string with velocity v. If it comes to rest at a distance a from the point, after describing a distance s, $v^2 = 2\mu g s$.

(39) A particle descends a rough circular tube from the extremity of the horizontal diameter. If it stops at the lowest point, shew that

$$3\mu\epsilon^{-\mu\pi} + 2\mu^2 = 1.$$

(40) If a particle under no forces be projected with velocity V along the inner surface of a rough sphere, determine the motion, and shew that it will return to the point of projection in the time

$$\frac{r}{\mu V}(\epsilon^{2\mu\pi} - 1),$$

where r is the radius of the sphere.

(41) A particle is attached to a smooth string which passes over a rough circular arc in a vertical plane; the particle initially at the extremity of a horizontal diameter is drawn up with constant acceleration $\frac{g}{\pi}$: shew that the

work expended in drawing it to the vertex of the circle is

$$Wa\left(\frac{3}{2}+\mu-\mu\frac{\pi}{4}\right),$$

where W is the weight of the particle, a the radius of the circle, and μ the coefficient of friction.

(42) A rough wire in the form of an equiangular spiral, whose angle is $\cot^{-1}2\mu$, is placed with its plane vertical and a particle slides down it under gravity, coming to rest at its lowest point; prove that at the starting-point the tangent makes with the horizon an angle $2\tan^{-1}\mu$, and that the velocity is greatest when the angle ϕ which the direction of motion makes with the horizon is given by the equation

$$(2\mu^2-1)\sin\phi+3\mu\cos\phi=2\mu.$$

(43) A particle falling under gravity down a rough cycloidal arc whose axis is vertical comes to rest at the lowest point: prove that if ϕ be the angle which the tangent at the starting-point makes with the horizon, then

$$\mu e^{\mu\phi}=\sin\phi-\mu\cos\phi.$$

(44) Two equal particles attracting each other with intensity the acceleration of which is $\rho^2\times$ distance are placed in two rough straight tubes at right angles to one another, and the friction is equal to the pressure in each tube; prove that if they be initially at unequal distances, one moves for a time $\frac{\pi}{2\rho}$ before the other begins to move, and that while they are approaching the point of intersection of the tubes they move in the same manner as the projections of the two extremities of a diameter of a circle upon a straight line on which the circle rolls.

(45) A particle moves on a rough curve under forces T in the tangent and N in the normal, prove that the velocity at any point is given by

$$\frac{1}{2}v^2e^{2\mu\psi}=\int(T+\mu N)\,e^{2\mu\psi}\,ds.$$

(46) A circular tube of small bore revolves with constant angular velocity ω about a vertical diameter, and a particle in it is projected from the lowest point with velocity due to the level of the highest point. Determine the motion, and shew that it is at its greatest distance from the axis after a time

$$\left(\frac{g}{a}+\omega^2\right)^{-\frac{1}{2}} \log_e \sqrt{\frac{\sqrt{\left(\frac{2g}{a}+\omega^2\right)}+\sqrt{\left(\frac{g}{a}+\omega^2\right)}}{\sqrt{\left(\frac{2g}{a}+\omega^2\right)}-\sqrt{\left(\frac{g}{a}+\omega^2\right)}}};$$

where a is the radius of the tube.

(47) A particle P, attached by a string of given length a, to a point S in a fixed axis SA, is attracted with constant intensity g in a direction always parallel to a line SB, which is inclined at a given angle to the axis SA, and revolves about it with a given angular velocity ω : shew that if $V=$ the velocity of P, $\omega'=$ the angular velocity of the plane PSA about SA, $\phi=\angle PSB$, $\theta=\angle PSA$,

$$\tfrac{1}{2}V^2=ga\cos\phi+a^2\omega\omega'\sin^2\theta+\text{const.}$$

Shew also that the dynamical conditions of this Problem are the same as those of a ball-pendulum under gravity, when the Earth's rotation is taken into account.

(48) A smooth circular tube is fixed at one point A and contains a particle which is initially at rest at the opposite end of the diameter through A. The tube is then made to revolve with constant angular velocity ω about an axis through A perpendicular to the plane of the tube. Prove that the angle described in the time t by the particle about the centre of the tube is

$$4\tan^{-1}\frac{e^{\omega t}-1}{e^{\omega t}+1}.$$

(49) A ring slides on a smooth elliptic wire which moves with constant angular velocity about an axis through its centre perpendicular to its plane. Determine the motion; and find the time of a small oscillation about the position of equilibrium where this is possible.

(50) If a particle slide along a smooth curve, which turns with constant angular velocity ω about an axis perpendicular to its plane passing through a fixed point O, then the velocity of the particle relatively to the moving curve is given by the equation

$$v^2 = c^2 + \omega^2 r^2,$$

where r is the distance of the particle from the point O; and the pressure on the curve will be given by the formula

$$R = \frac{v^2}{\rho} + 2\omega v + \omega^2 p,$$

where p is the perpendicular from O on the tangent.

(51) If a curve revolve with constant angular velocity about a vertical axis in its plane, prove that the time of a small oscillation of a particle sliding on the curve about its position of relative equilibrium is

$$\frac{2\pi}{\omega}\sqrt{\frac{\rho \sin \alpha}{r - \rho \sin \alpha \cos^2 \alpha}},$$

ρ being the radius of curvature at the point of equilibrium, α the angle made by the normal at that point with the vertical, r the distance of the point from the axis of revolution, and ω the angular velocity of the curve.

(52) A fine straight tube rotates in a plane with constant angular velocity ω about a point in its length while the plane rotates with constant angular velocity ω' about a horizontal axis through that point, prove that the equation of motion of a particle placed in the tube is

$$\frac{d^2 r}{dt^2} - (\omega^2 + \omega'^2 \cos^2 \omega t)\, r = g \sin \omega' t \cos \omega t,$$

the tube being initially perpendicular to the horizontal axis and the plane horizontal.

(53) AB is the diameter of a sphere of radius a; a centre of attraction at A attracts with intensity ($\mu \times$ distance); from

the extremity of a diameter perpendicular to AB a particle is projected in any direction along the inner surface with a velocity $(2\mu)^{\frac{1}{2}}a$: shew that the velocity of the particle at any point P is $2\sqrt{\mu}\, a \sin\theta$, and the pressure is $\mu a\,(3\sin^2\theta - 1)$, where θ is the angle PAB, so long as the particle remains in contact with the surface.

(54) A particle is attached by a fine string to the apex of a right vertical cone whose semivertical angle is β, and is projected from a position of rest on the cone with an initial angular velocity ω (about its axis) which is less than Ω, the least angular velocity which would make the particle leave the cone. If the coefficient of friction between the particle and cone be μ, find the position of the particle and the tension of the string at a given instant; and shew that it will come to rest after a time

$$\frac{1}{2\mu\Omega\cos\beta}\log\frac{\Omega+\omega}{\Omega-\omega}.$$

(55) If a particle be projected horizontally inside a right circular cone of vertical angle 2α whose axis is vertical and vertex downwards with the velocity which it would acquire by sliding down a generator from the point of projection to the vertex, shew that:

1. The motion is oscillatory, and that the particle never descends lower than its initial position.

2. The curve traced out by the particle on the cone if developed into a plane has its equation

$$\rho^3\left(\frac{dr}{d\theta}\right)^2 = r^2\,(r-\rho)\,(\rho^2+\rho r - r^2),$$

where r is the distance of the particle from the vertex, ρ its initial distance.

3. The reaction is given by the equation

$$R = mg\left\{\sin\alpha + 2\frac{\cos^2\alpha}{\sin\alpha}\left(\frac{\rho}{r}\right)^3\right\}.$$

(56) A particle is in equilibrium on the surface of a smooth thin hemispherical bowl which attracts according to the law of nature. If it be slightly displaced, shew that the time of a small oscillation is

$$4\pi\sqrt{\frac{\sqrt{2}a^3}{M}},$$

where a is the radius of the bowl, and M the mass.

(57) If a particle be projected inside a smooth paraboloid of revolution, axis vertical and vertex downwards, horizontally at the level of the focus with velocity $\sqrt{2nag}$, the initial radius of curvature of the path will be

$$\frac{2\sqrt{2na}}{\sqrt{n^2+1}}.$$

(58) A particle is projected in a smooth paraboloid whose axis is vertical so as very nearly to describe a circle whose diameter is the latus rectum of the generating parabola; prove that the time of a small oscillation is the same as that of a pendulum of length a, where $4a$ is the latus rectum.

(59) A particle moves in the interior of a smooth paraboloid of revolution whose axis is vertical and vertex downwards. Shew that the differential equation of its path on a horizontal plane is

$$4a^2u\left(\frac{d^2u}{d\theta^2}+u\right)+\frac{d}{d\theta}\left(\frac{1}{u}\cdot\frac{du}{d\theta}\right)=2\frac{ag}{h^2u^2},$$

where $4a$ is the latus rectum of the generating parabola.

(60) A particle under an attraction varying inversely as the cube of the distance from a given plane, is constrained to move on a smooth spherical surface, having been projected with the velocity due to an infinite distance; prove that the resultant acceleration of the particle always passes through a fixed point.

(61) A particle is attached to the highest point of a smooth fixed sphere of radius a, by an elastic string whose natural length is $\frac{\pi a}{8}$, and the weight of the particle is sufficient to stretch the string to double its natural length; at first the particle moves with constant velocity in a small circle, the string being stretched to double its natural length; prove that if the motion be slightly disturbed the time of a small oscillation will be

$$2\pi\sqrt{\frac{a}{g}\frac{\pi\sqrt{2}}{(4\sqrt{2}-5)\pi+8\sqrt{2}}},$$

and find the greatest impulse it can receive along the direction of the string without leaving the sphere.

(62) A rough paraboloid of revolution of latus rectum $4a$ and coefficient of friction $\cot\beta$ revolves with constant angular velocity about its axis which is vertical; prove that for any given angular velocity greater than

$$\sqrt{\frac{g}{2a}\cot\frac{\beta}{2}}$$

or less than

$$\sqrt{\frac{g}{2a}\tan\frac{\beta}{2}}$$

a particle can rest anywhere on the surface except within a certain zone, but that for any intermediate angular velocity equilibrium is possible at every point of the surface.

(63) An anchor ring is formed by the revolution of a circle of radius c about an axis in its own plane at a distance a from the centre. A particle is projected along the smaller equator with velocity v and is under an attraction to the centre of the axis μr^n. If the particle be slightly displaced, prove that it will cut its original path at equal angular intervals α, where

$$\left(\frac{\pi}{\alpha}\right)^2 = \left\{\frac{a-c}{c}\frac{\mu a(a-c)^n}{v^2} - 1\right\}.$$

(64) A smooth surface is generated by the revolution of the curve $x^2y = c^3$ about the axis of y which is directed vertically downwards, and a particle under gravity is projected along the surface with velocity due to the level of the horizontal plane through the origin.

Prove that its path will intersect all the meridians at a constant angle.

(65) Find the form of the smooth surface on which if a particle move, under the action of a central force varying, according to a given law, with the distance, the normal pressure shall be constant.

(66) Find the Brachistochrone of continuous plane curvature, when the velocity is always proportional to the curvature.

(67) Find the Brachistochrone in a medium in which the velocity depends upon the direction of motion only.

Take particular cases: such as

a) $v = A\lambda + B\mu + C\nu$,

b) $v = A\lambda^2 + B\mu^2 + C\nu^2$,

c) $v = A\mu\nu + B\nu\lambda + C\lambda\mu$;

λ, μ, ν being direction-cosines, and A, B, C constants.

CHAPTER VII.

MOTION IN A RESISTING MEDIUM.

207. WHEN a body moves in a fluid, whether liquid or gaseous, it must, in displacing the particles of the medium and in rubbing against them, lose part of its own velocity. The resistance of a fluid to a body moving in it produces therefore a retardation; but, in consequence of the great difficulty of making accurate experiments on the subject, the laws of the resistance of fluids have not yet been satisfactorily ascertained.

For a velocity neither very great nor very small, the general approximate law seems to be, that the resistance to a plane surface, moving with its plane at right angles to the line of motion, varies as the extent of the surface, the density of the resisting medium, and the square of the velocity taken conjointly; and the retardation due to the resistance is therefore equal to the numerical value of the resistance divided by the number of units of mass in the body.

It is well that the student should observe that in all cases of resistance, as in those of friction, the motives are essentially irreversible. We are no longer dealing with *conservative* systems of motion, so long at least as we confine our attention to the motion of the resisted body alone.

208. *A particle under no forces is projected in a resisting medium of uniform density, of which the resistance varies as the* n^{th} *power of the velocity; to determine the motion.*

The motion will evidently be rectilinear. Let s be the distance of the particle from a fixed point in the line of motion at the time t, v its velocity at that time. The

retardation due to the resistance may be represented by kv^n, k being a constant, and the equation of motion is

$$\frac{dv}{dt} = -kv^n \qquad\text{(1)},$$

or

$$v\frac{dv}{ds} = -kv^n \qquad\text{(2)}.$$

Therefore

$$\frac{1}{v^n}\frac{dv}{dt} = -k,$$

$$\frac{1}{v^{n-1}}\frac{dv}{ds} = -k.$$

Integrating, supposing the initial velocity V,

$$\frac{1}{v^{n-1}} - \frac{1}{V^{n-1}} = (n-1)\,kt \qquad\text{(3)},$$

$$\frac{1}{v^{n-2}} - \frac{1}{V^{n-2}} = (n-2)\,ks \qquad\text{(4)},$$

and the elimination of v between (3) and (4) will give s in terms of t.

We see from (3) and (4) that if $n > 1$, the velocity never vanishes; and that if $n > 2$, the distance gone increases indefinitely.

209. The Rev. F. Bashforth, *Motion of Projectiles*, found that for small variations of velocity we might put $n = 3$.

If d = diameter of shot in inches, w = number of pounds in the shot, then the retardation due to the resistance was put $= 10^{-9}\frac{d^2}{w}Kv^3$, so that $k = 10^{-9}\frac{d^2}{w}K$, and K was determined by experiment for velocities proceeding by increments of 10 between 900 and 1700 feet per second, K attaining its maximum value for a velocity of about 1200.

The numerical values of K for elongated and spherical projectiles are given in Tables I. and II. in the "Motion of Projectiles."

Tables also were calculated by Mr Bashforth from formulæ (3) and (4) (Tables VIII.—XI.), giving $\frac{d^2}{w}s$ and $\frac{d^2}{w}t$ for every decrement of 10 in the velocity between 1700 and 900, using the mean value of K between each pair of velocities, and from these tables we can determine s in terms of v and t in terms of v for any shot, neglecting gravity, and consequently s in terms of t.

210. There is one case in which the above solution fails, namely when $n = 1$, or the resistance varies as the velocity. In this case k is the reciprocal of a time and may be put $= \frac{1}{\tau}$, and then

$$\frac{dv}{dt} = -\frac{v}{\tau} \quad\text{.........................(1)},$$

or

$$\frac{1}{v}\frac{dv}{dt} = -\frac{1}{\tau} \quad\text{.........................(2)}.$$

Hence

$$\log\frac{V}{v} = \frac{t}{\tau} \quad\text{.........................(3)};$$

and therefore

$$v = \frac{ds}{dt} = V\epsilon^{-\frac{t}{\tau}}.$$

Integrating, we have

$$s = V\tau\left(1 - \epsilon^{-\frac{t}{\tau}}\right) \quad\text{.................(4)}.$$

Equations (3) and (4) determine the velocity and the position of the particle at any instant. They shew that the velocity continually diminishes without ever actually becoming zero, but that the distance passed over by the particle has a definite limit, for when

$$t = \infty, \quad s = V\tau.$$

211. *A particle, under a constant force in its line of motion, moves in a resisting medium of uniform density, of which the resistance varies as the square of the velocity; to determine the motion.*

Suppose the particle projected from the origin with the velocity V, and let v be its velocity at any time t, x its distance from the origin at that time, and f the constant acceleration due to the force.

Assume K to be the velocity with which the particle would have to be animated that the retardation due to the resistance might be equal to f, then the retardation when the velocity is v may be represented by $f\dfrac{v^2}{K^2}$.

Let f act so as to diminish x; then the equation of motion is

$$\frac{dv}{dt} = -\frac{f}{K^2}(K^2 + v^2)$$

or

$$v\frac{dv}{dx} = -\frac{f}{K^2}(K^2 + v^2).$$

Integrating, and determining the constants so that when

$$x = 0, \quad t = 0, \quad v = V,$$

we obtain

$$\frac{ft}{K} = \tan^{-1}\frac{V}{K} - \tan^{-1}\frac{v}{K} = \tan^{-1}\frac{K(V-v)}{K^2 + Vv},$$

$$\frac{2fx}{K^2} = \log\frac{K^2 + V^2}{K^2 + v^2}.$$

Let T be the time at which the velocity becomes zero, and h the corresponding value of x, then

$$T = \frac{K}{f}\tan^{-1}\frac{V}{K}, \text{ and } h = \frac{K^2}{2f}\log\left(1 + \frac{V^2}{K^2}\right).$$

After this the particle begins to return, the resistance therefore tends to increase x, and the equation of motion is

$$-\frac{dv}{dt} = -\frac{f}{K^2}(K^2 - v^2),$$

or
$$v\frac{dv}{dx} = -\frac{f}{K^2}(K^2 - v^2).$$

Integrating, and determining the constants so that when

$$v = 0,\ x = h,\ t = T,$$

we obtain

$$\frac{2f}{K}(t - T) = \log\frac{K+v}{K-v},$$

$$\frac{2f}{K^2}(h - x) = \log\frac{K^2}{K^2 - v^2}.$$

Let U be the velocity with which the particle will return to the point of projection; then, putting $x = 0$ in the latter equation, we obtain

$$\frac{U^2}{K^2} = 1 - \epsilon^{-\frac{2fh}{K^2}};$$

or, substituting for h its value,

$$\frac{U^2}{K^2} = \frac{\dfrac{V^2}{K^2}}{1 + \dfrac{V^2}{K^2}},$$

whence

$$\frac{1}{U^2} - \frac{1}{V^2} = \frac{1}{K^2}.$$

This shews, as we might expect, that the particle returns to the point of projection with diminished velocity.

212. The results of the last Proposition are applicable to bodies projected in a resisting medium vertically upwards or downwards under gravity; for the acceleration due to gravity

may still be considered constant, although not the same as for a particle in vacuo. The effective attraction of gravity is in fact the difference of the weights of the body and the fluid displaced, so that if α be the ratio of the density of the fluid displaced to that of the body, effective gravity

$$=W(1-\alpha)=Mg(1-\alpha),$$

where W and M are the weight and mass of the body, and therefore the acceleration caused by gravity $=g(1-\alpha)$. By substituting this for f in the results of § 211, we may obtain formulæ for the motion of bodies in a vertical direction under gravity. Hailstones and raindrops afford a good illustration of the *Terminal Velocity* indicated by the result of § 211.

213. *To find the equations of motion, in a resisting medium, of a particle under any forces.*

Let x, y, z be the co-ordinates of the particle relative to an assumed system of rectangular axes, at the time t, and let X, Y, Z be the component accelerations, parallel to the axes, due to the forces acting on the particle. Then denoting by R the retardation due to the resistance, which lies in the tangent to the path described, and in a direction opposed to the motion, we have

$$\frac{d^2x}{dt^2}=X-R\frac{dx}{ds},$$

$$\frac{d^2y}{dt^2}=Y-R\frac{dy}{ds},$$

$$\frac{d^2z}{dt^2}=Z-R\frac{dz}{ds}.$$

These are the general equations of motion. In any particular case R will be given as a function of the density of the medium and the velocity of the particle, and particular methods will be necessary for obtaining the path of the particle and its position at any time. These equations will enable us, when X, Y, Z are given, to determine the resistance that a given path may be described.

214. *A particle under gravity is projected from a given point in a given direction with a given velocity, and moves in a uniform medium whose resistance varies as some power of the velocity; to determine the motion.*

Take the given point as origin, the axis of x horizontal, the axis of y vertically upwards, so that the plane of xy may contain the direction of projection; let g denote the acceleration of gravity, v the velocity of the particle at any point, u its horizontal component, ϕ the inclination of the direction of motion to the horizon, and $R = kv^n$ the retardation due to the resistance.

Then the equations of motion are, resolving horizontally and vertically,

$$\frac{d^2x}{dt^2} = -R\frac{dx}{ds} \quad\text{.......................(1)},$$

$$\frac{d^2y}{dt^2} = -g - R\frac{dy}{ds} \quad\text{.....................(2)},$$

or, resolving in the direction of the tangent and normal,

$$\frac{d^2s}{dt^2} = -g\sin\phi - R \quad\text{..................(3)},$$

$$\frac{v^2}{\rho} = g\cos\phi \quad\text{.............................(4)}.$$

Since $v = \dfrac{ds}{dt}$, $u = v\cos\phi$ and $\rho = -\dfrac{ds}{d\phi}$, equations (1) and (4) may be written

$$\frac{du}{dt} = -R\cos\phi \quad\text{.......................(5)},$$

$$v\frac{d\phi}{dt} = -g\cos\phi \quad\text{.....................(6)},$$

and therefore

$$\begin{aligned}\frac{du}{d\phi} &= \frac{Rv}{g} \\ &= \frac{kv^{n+1}}{g} \\ &= \frac{k}{g}u^{n+1}\sec^{n+1}\phi \quad\text{.............(7)}.\end{aligned}$$

Integrating this equation, denoting by u_0 the velocity at the vertex of the trajectory,

$$\frac{1}{u_0^n} - \frac{1}{u^n} = \frac{k}{g} P_\phi, \qquad\qquad (8),$$

if P_ϕ denote the integral $\int_0 n \sec^{n+1} \phi d\phi$.

Therefore $$u = \frac{u_0}{(1 - \gamma P_\phi)^{\frac{1}{n}}},$$

where $$\gamma = \frac{ku_0^n}{g} = \frac{\text{retardation at vertex}}{\text{acceleration of gravity}}$$

$$= \frac{\text{resistance at vertex}}{\text{weight of shot}}.$$

From equation (6)

$$\frac{dt}{d\phi} = -\frac{v}{g} \sec \phi = -\frac{u}{g} \sec^2 \phi,$$

therefore, if α be the angle of projection,

$$t = \frac{u_0}{g} \int_\phi^\alpha \frac{\sec^2 \phi d\phi}{(1 - \gamma P_\phi)^{\frac{1}{n}}} \qquad\qquad (9).$$

Again, $$x = \int_0 u dt = \int_\phi^\alpha \frac{u^2}{g} \sec^2 \phi d\phi$$

$$= \frac{u_0^2}{g} \int_\phi^\alpha \frac{\sec^2 \phi d\phi}{(1 - \gamma P_\phi)^{\frac{2}{n}}} \qquad\qquad (10),$$

and $$y = \frac{u_0^2}{g} \int_\phi^\alpha \frac{\tan \phi \sec^2 \phi d\phi}{(1 - \gamma P_\phi)^{\frac{2}{n}}} \qquad\qquad (11).$$

Equations (9), (10), (11) give t, x, y in terms of ϕ.

For $$n = 3, \quad P = 3 \tan \phi + \tan^3 \phi,$$

and the integrals in (9), (10), (11) were calculated by quadratures for different values of γ and for certain ranges of angle, and the nominal values tabulated, in Tables IV. V. VI. in Mr Bashforth's "Motion of Projectiles."

215. For $n=1$, putting $R=\frac{v}{\tau}$, then τ is the measure of a time, and

$$P=\int_0 \sec^2\phi\, d\phi = \tan\phi,$$

$$t=\frac{u_0}{g}\int_\phi^\alpha \frac{\sec^2\phi\, d\phi}{1-\gamma\tan\phi}$$

$$=\frac{u_0}{g\gamma}\log\frac{1-\gamma\tan\phi}{1-\gamma\tan\alpha} \quad\text{.....................(1)};$$

and since $\qquad \gamma=\frac{u_0}{g\tau}$,

$$t=\tau\log\frac{1-\gamma\tan\phi}{1-\gamma\tan\alpha};$$

$$x=\frac{u_0^2}{g}\int_\phi^\alpha \frac{\sec^2\phi\, d\phi}{(1-\gamma\tan\phi)^2}$$

$$=u_0\tau\left(\frac{1}{1-\gamma\tan\alpha}-\frac{1}{1-\gamma\tan\phi}\right) \quad\text{... (2)};$$

$$y=\frac{u_0^2}{g}\int_\phi^\alpha \frac{\sec^2\phi\tan\phi\, d\phi}{(1-\gamma\tan\phi)^2}$$

$$=g\tau^2\left(\frac{1}{1-\gamma\tan\alpha}-\frac{1}{1-\gamma\tan\phi}-\log\frac{1-\gamma\tan\phi}{1-\gamma\tan\alpha}\right)\text{...(3)};$$

and the elimination of $\tan\phi$ between (2) and (3) will give the relation between x and y.

Or immediately, resolving horizontally and vertically,

$$\frac{d^2x}{dt^2}=-\frac{1}{\tau}\frac{dx}{dt} \quad\text{..........................(4)},$$

$$\frac{d^2y}{dt^2}=-\frac{1}{\tau}\frac{dy}{dt}-g \quad\text{.....................(5)};$$

and integrating, supposing V the velocity and α the angle of projection,

$$\frac{dx}{dt} = V\cos\alpha e^{-\frac{t}{\tau}},$$

$$\frac{dy}{dt} + g\tau = (V\sin\alpha + g\tau)e^{-\frac{t}{\tau}},$$

and integrating again,

$$x = V\tau\cos\alpha\,(1 - e^{-\frac{t}{\tau}}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$y + gt\tau = (V\tau\sin\alpha + g\tau^2)(1 - e^{-\frac{t}{\tau}}) \ldots\ldots\ldots\ldots(7).$$

Eliminating t between (6) and (7),

$$y + g\tau^2\log\frac{V\tau\cos\alpha}{V\tau\cos\alpha - x} = \left(\tan\alpha + \frac{g\tau}{V\cos\alpha}\right)x,$$

the equation of the trajectory.

Differentiating this equation twice, we obtain

$$\frac{d^2y}{dx^2} + \frac{g\tau^2}{(V\tau\cos\alpha - x)^2} = 0,$$

the differential equation of the trajectory.

216. For $n = 2$, putting $R = \frac{v^2}{a}$, then a is the measure of a length, and putting $p = \tan\phi$,

$$P = \int_0 2\sec^3\phi\,d\phi = 2\int\sqrt{1+p^2}\,dp$$

$$= p\sqrt{1+p^2} + \log(p + \sqrt{1+p^2}).$$

The equations of motion are, resolving horizontally and vertically,

$$\frac{d^2x}{dt^2} = -\frac{1}{a}\left(\frac{ds}{dt}\right)^2\frac{dx}{ds} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{d^2y}{dt^2} = -\frac{1}{a}\left(\frac{ds}{dt}\right)^2\frac{dy}{ds} - g \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Equation (1) may be written

$$\frac{du}{dt} = -\frac{1}{a}\frac{ds}{dt}u,$$

or

$$\frac{1}{u}\frac{du}{dt} = -\frac{1}{a}\frac{ds}{dt};$$

and integrating,

$$u = V\cos\alpha e^{-\frac{s}{a}} \qquad \text{.........................(3).}$$

From equation (8) of § 214,

$$\frac{1}{u^2} - \frac{1}{V^2\cos^2\alpha} = \frac{1}{ga}(P_\alpha - P_\phi),$$

or

$$e^{2\frac{s}{a}} - 1 = \frac{V^2\cos^2\alpha}{ga}\left(\tan\alpha\sec\alpha - \tan\phi\sec\phi + \log\frac{\tan\alpha + \sec\alpha}{\tan\phi + \sec\phi}\right) \qquad \text{............ (4),}$$

the intrinsic equation of the path.

Differentiating this equation with respect to x,

$$\frac{2}{a}e^{2\frac{s}{a}}\frac{ds}{dx} = -\frac{V^2\cos^2\alpha}{ga}\frac{dP}{dx} = -\frac{2V^2\cos^2\alpha}{ga}\sqrt{1+p^2}\frac{dp}{dx},$$

or

$$\frac{dp}{dx} + \frac{g}{V^2\cos^2\alpha}e^{2\frac{s}{a}} = 0 \qquad \text{..............(5),}$$

the differential equation of the path.

If S, s denote the arcs of the trajectory in a non-resisting and a resisting medium, measured from the point of projection to any two points at which the tangents are parallel; then, since in the non-resisting medium $a = \infty$, therefore

$$\frac{dp}{ds} = -\frac{g}{V^2\cos^2\alpha}e^{2\frac{s}{a}}\cos\phi,$$

$$\frac{dp}{dS} = -\frac{g}{V^2\cos^2\alpha}\cos\phi,$$

and

$$\frac{dS}{ds} = e^{2\frac{s}{a}},$$

and integrating,

$$\frac{S}{a} = \frac{1}{2}(e^{2\frac{s}{a}} - 1) \text{..................} (6).$$

217. For a flat trajectory, p being always small, we may put $\frac{ds}{dx} = 1$, and then equation (5) may be written

$$\frac{dp}{dx} + \frac{g}{V^2 \cos^2 \alpha} e^{2\frac{s}{a}} \frac{ds}{dx} = 0.$$

Integrating,

$$p + \frac{ga}{2V^2 \cos^2 \alpha} e^{2\frac{s}{a}} = \tan \alpha + \frac{ga}{2V^2 \cos^2 \alpha};$$

or

$$e^{2\frac{s}{a}} - 1 = \frac{2V^2 \cos^2 \alpha}{ga} (\tan \alpha - p).$$

And substituting again in equation (5)

$$\frac{dp}{dx} - \frac{2p}{a} = -\frac{g}{V^2 \cos^2 \alpha} - \frac{2}{a} \tan \alpha.$$

Multiplying by $e^{-2\frac{x}{a}}$ and integrating,

$$pe^{-2\frac{x}{a}} = -\frac{ga}{2V^2 \cos^2 \alpha} + \left(\frac{ga}{2V^2 \cos^2 \alpha} + \tan \alpha\right) e^{-2\frac{x}{a}},$$

or

$$p = \frac{dy}{dx} = -\frac{ga}{2V^2 \cos^2 \alpha} e^{2\frac{x}{a}} + \frac{ga}{2V^2 \cos^2 \alpha} + \tan \alpha.$$

And integrating again,

$$y = x\left(\tan \alpha + \frac{ga}{2V^2 \cos^2 \alpha}\right) - \frac{ga^2}{4V^2 \cos^2 \alpha} (e^{2\frac{x}{a}} - 1).$$

Expanding $e^{2\frac{x}{a}}$ in ascending powers of $\frac{x}{a}$, a being supposed large,

$$y = x \tan \alpha - \frac{gx^2}{2V^2 \cos^2 \alpha} - \frac{gx^3}{3aV^2 \cos^2 \alpha} \cdots\cdots,$$

of which the first two terms will represent the trajectory in a non-resisting medium.

218. *A particle moves in a resisting medium under a central attraction; to determine the orbit.*

Let P be the acceleration due to the central attraction, R the retardation due to the resistance of the medium; then resolving along and perpendicular to the radius vector,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P - R\frac{dr}{ds} \quad \text{..............(1)},$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -R\frac{rd\theta}{ds} \quad \text{.....................(2)}.$$

Putting $r^2\dfrac{d\theta}{dt} = h$, equation (2) may be written

$$\frac{dh}{dt} = -Rr^2\frac{d\theta}{ds},$$

or

$$\frac{1}{h}\frac{dh}{dt} = -\frac{R}{v};$$

and therefore

$$h = h_0 e^{-\int \frac{R}{v}dt} \quad \text{...................... (3)}.$$

Or the equation may be written

$$\frac{1}{h}\frac{dh}{ds} = -\frac{R}{v^2};$$

and therefore

$$h = h_0 e^{-\int \frac{R}{v^2}ds} \quad \text{......................(4)}.$$

Again, putting $r = \dfrac{1}{u}$, we have $\dfrac{d\theta}{dt} = hu^2$,

and

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta},$$

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} - \frac{dh}{dt}\frac{du}{d\theta}$$

$$= -h^2u^2\frac{d^2u}{d\theta^2} + h\frac{R}{v}\frac{du}{d\theta}$$

$$= -h^2u^2\frac{d^2u}{d\theta^2} + R\frac{1}{u^2}\frac{du}{ds};$$

and therefore

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -h^2u^2\frac{d^2u}{d\theta^2} - R\frac{dr}{ds} - h^2u^3$$

$$= -P - R\frac{dr}{ds};$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} \qquad\qquad (5),$$

an equation of the same form as that for the motion in a non-resisting medium, h however being now variable.

219. If in addition to the central attraction, there is a transversal force producing acceleration T, we shall obtain the equation analogous to (5) most simply by resolving in the normal, and then

$$\frac{v^2}{\rho} = P\sin\phi + T\cos\phi,$$

where

$$\tan\phi = \frac{rd\theta}{dr} = -\frac{ud\theta}{du}.$$

Therefore

$$P + T\cot\phi = P - T\frac{du}{ud\theta}$$

$$= \frac{v^2}{\rho\sin\phi} = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right),$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} - \frac{T}{h^2u^3}\frac{du}{d\theta},$$

an equation of the same form as that obtained in § 136.

EXAMPLES.

(1) If the time is a quadratic function of the length traversed, prove that the resistance varies as the cube of the velocity.

(2) Shew that the solution of the differential equation for vibrations resisted by friction proportional to the velocity, but otherwise free, viz.

$$\ddot{u} + k\dot{u} + n^2u = 0,$$

may be put into the form

$$u = e^{-\frac{1}{2}kt}\left\{\dot{u}_0 \frac{\sin n't}{n'} + u_0\left(\cos n't + \frac{k}{2n'}\sin n't\right)\right\},$$

where $n'^2 = n^2 - \frac{1}{4}k^2$, and $\dot{u}_0$, u_0 are the values of the velocity and displacement when $t = 0$.

Deduce the complete solution of

$$\ddot{u} + k\dot{u} + n^2u = U,$$

in the form

$$u = e^{-\frac{1}{2}kt}\left\{\dot{u}_0 \frac{\sin n't}{n'} + u_0\left(\cos n't + \frac{k}{2n'}\sin n't\right)\right\}$$

$$+ \frac{1}{n'}\int_0^t e^{-\frac{1}{2}k(t-t')}\sin n'(t-t')\, U'\, dt,$$

where U' is the same function of t' as U is of t.

(3) Determine the motion of a body under an attraction towards a fixed centre proportional to the distance in a medium whose resistance is proportional to the velocity.

A body performs rectilinear vibrations in this medium in a period T, and the co-ordinates of the extremities of three consecutive semivibrations are a, b, c; prove that the co-ordinate of the position of equilibrium and the time of vibration if there were no resistance are respectively

$$\frac{ac-b^2}{a+c-2b} \text{ and } T\left\{1+\frac{1}{\pi^2}\left(\log\frac{a-b}{c-b}\right)^2\right\}^{-\frac{1}{2}}.$$

(4) If chords be drawn from either extremity of a vertical diameter of a circle, the time of descent down each of them in a medium whose resistance varies as the velocity is the same.

(5) One particle begins to fall from the higher extremity of a vertical line, and at the same instant another is projected upwards from the other extremity with a given velocity, the particles moving in a medium of which the resistance varies as the velocity; shew that the time at which they will meet will be $\tau \log \dfrac{V\tau}{V\tau - a}$, where a is the length of the line, V the velocity of projection, and the retardation due to the resistance is $\dfrac{1}{\tau}$ of the velocity.

(6) A light elastic string whose unstretched length is a is fastened at one end and to the other end is attached a particle, which hanging freely would stretch the string to a length $2a$. The particle is projected vertically upwards from the point at which the string is fastened in a medium of resistance producing retardation $\dfrac{v^2}{2a}$. If h be the height attained, U the velocity of projection, V the velocity with which the particle returns to the point of projection,

$$\frac{1}{2}U^2 = g\left\{a(e-1)+(h-a)e^{\frac{h}{a}}\right\},$$

$$\frac{1}{2}V^2 = g\left\{a\left(1+\frac{1}{e}\right)-(h+a)e^{-\frac{h}{a}}\right\}$$

(7) Determine the law of attraction that a particle may always descend to a given centre in the same time from whatever distance it commences its motion, the medium in which the particle moves being uniform, and the resistance varying as the square of the velocity.

(8) If one particle be projected in a medium, the resistance of which varies as the velocity, and another be projected in vacuo at the same angle, and with the same velocity, both particles being under gravity, and if t_1, t_2 be the times of describing two arcs in the medium and in vacuo so related to each other that the tangents at their extremities shall be parallel to each other, then

$$\epsilon^{\frac{t_1}{\tau}} - 1 = \frac{t_2}{\tau}.$$

(9) Prove the following equations applicable to the motion of a shot resisted by the air with retardation $f(v)$, v being the velocity and ψ the inclination to the horizon of the direction of motion:

$$\frac{dv}{d\psi}\cos\psi - v\sin\psi = \frac{v}{g}f(v),$$

$$v\frac{d\psi}{dt} = -g\cos\psi.$$

Prove also that if ψ_0 is the initial value of ψ, and t, x, y the time and horizontal and vertical distances from the point of projection,

$$gt = \int_{\psi}^{\psi_0} v\sec\psi d\psi,$$

$$gx = \int_{\psi}^{\psi_0} v^2\, d\psi,$$

$$gy = \int_{\psi}^{\psi_0} v^2\tan\psi d\psi.$$

Solve completely the case for which

$$f(v) = \frac{v}{\tau}.$$

(10) If the horizontal distance of a projectile in a resisting medium from the point of projection be connected with the time by the equation $x = f(t)$, prove that the equation of the trajectory is

$$y = -gf(t)\int\frac{dt}{f'(t)} + g\int\frac{f(t)}{f'(t)}\,dt + Af(t) + B,$$

where A and B are constants.

In the case when $t = ax + bx^2$, shew that the equation of the trajectory is

$$y = x\tan\alpha - g\left(\frac{1}{2}a^2x^2 + \frac{2}{3}abx^3 + \frac{1}{3}b^2x^4\right).$$

(11) A particle moves under gravity in a medium in which the resistance varies as the n^{th} power of the velocity, V_1, V_2 being its velocities at the two points where its direction of motion makes an angle ϕ with the horizon, and V its velocity at the highest point; prove that

$$\frac{1}{V_1^n} + \frac{1}{V_2^n} = \frac{2\cos^n\phi}{V^n}.$$

(12) If the resistance vary as the n^{th} power of the velocity, and if f be the retardation due to the resistance when a shot is ascending at an inclination ϕ, f_0 when it is moving horizontally, and f' when it is descending at an inclination ϕ in the trajectory, prove that

$$\frac{1}{f'} + \frac{1}{f} = \frac{2\cos^n\phi}{f_0},$$

$$\frac{1}{f'} - \frac{1}{f} = \frac{2}{g}\cos^n\phi\int_0 n\sec^{n+1}\phi\,d\phi.$$

(13) A body of mass m is describing a parabola under gravity, and a tangential impulse mu acts on it. Prove that the focus of the new trajectory moves towards the body a distance $\dfrac{2v-u}{2g}u$, where v was the velocity of the body.

If the body is acted on by a uniform resistance we may conceive the path as the envelope of a system of parabolas. Apply the above to find the relation between corresponding arcs of the path and the locus of the foci of the enveloping parabolas.

(14) A particle of weight W moves under gravity in a medium of which the resistance R is always small and varies according to a given law; shew that the velocity of the focus of the instantaneous parabola at any time is $\frac{R}{W} \times$ velocity of the particle.

(15) Prove that the angular velocity of regression of the apse line of a planet P, moving in a medium producing retardation R, is

$$\frac{2R}{eV} \sin PSH,$$

where S is the sun, H the other focus of the orbit, e the eccentricity, and V the velocity.

(16) Explain how it is that a resisting medium, even though acting for a short time only, would accelerate the mean motion of a planet.

(17) A particle is moving amidst rays diverging from a point, which offer resistance only to motion across them with resistance proportional to the velocity. Shew that it is possible for the particle to move with constant angular velocity about the point, and find the path and the circumstances of projection.

(18) A particle describes an equiangular spiral under a central attraction in a medium of which the resistance varies as the square of the velocity.

Prove that the distance at which the attraction is a maximum is half the distance at which the velocity is a maximum, and that these distances are independent of the initial distance or initial velocity.

(19) The retardation due to the resistance of a medium being kv^4, prove that the orbit under a central attraction $\frac{\mu}{r^3}$ will be an equiangular spiral if the velocity of projection be that in a circle at the same distance, and the angle of projection be $\cos^{-1} 2\mu k$.

(20) Shew that the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + n^2x = n^2P\left(1 + 2\Sigma_1^\infty \cos ipt\right),$$

in which i has all positive integral values, and k is less than n, represents cycloidal pendulum motion, with viscous resistance, under the action of an infinite series of equal impulses (in the same direction) succeeding one another at intervals of $\frac{2\pi}{p}$.

Integrate this equation; and, by comparing the result with that obtained by treating the problem for each impulse separately from an epoch so distant that the motion has become independent of the initial circumstances, shew that

$$\frac{1}{n^2} + 2\Sigma_1^\infty \frac{(n^2 - i^2p^2)\cos ipt + 2ikp \sin ipt}{(n^2 - i^2p^2)^2 + 4i^2p^2k^2}$$

$$= \frac{2\pi}{pn_1}\epsilon^{-kt}\frac{\left(1 - \epsilon^{-\frac{2\pi k}{p}}\cos\frac{2\pi n_1}{p}\right)\sin n_1t + \epsilon^{-\frac{2\pi k}{p}}\sin\frac{2\pi n_1}{p}\cos n_1t}{1 - 2\epsilon^{-\frac{2\pi k}{p}}\cos\frac{2\pi n_1}{p} + \epsilon^{-\frac{4\pi k}{p}}};$$

where $n_1 = \sqrt{n^2 - k^2}$, and t lies between 0 and $\frac{2\pi}{p}$.

(21) Prove that the cycloid is still a tautochrone under gravity when the resistance varies as the velocity.

Prove that the same is true also of any tautochrone.

(22) The time of vibration from rest to rest of a cycloidal pendulum when unresisted being $\frac{\pi}{n}$, prove that if the resistance of the air produce retardation $2n \sin \alpha \times$ velocity, in order that the arc of oscillation may be constantly $2c$, each time the bob passes through the lowest point, it must receive an impulse in the direction of motion

$$mnce^{-\alpha \tan \alpha}\left(e^{\frac{\pi}{2}\tan \alpha} - e^{-\frac{\pi}{2}\tan \alpha}\right),$$

where m is the mass of the bob.

(23) A particle is projected in a medium the resistance of which produces retardation $\frac{3}{\tau} \times$ velocity, and is under an attraction to a fixed point which produces acceleration $\frac{2}{\tau^2} \times$ distance. Prove that the particle will describe a parabola, tending to come to rest at the origin.

(24) If a particle under a central attraction producing acceleration $\mu^2 r$ move in a medium of which the resistance produces retardation $2k$ (velocity), prove that it will describe the curve

$$\frac{1}{2}\log\frac{(ay-cx)^2+(by-dx)^2}{(ad-bc)^2} + \frac{k}{\sqrt{\mu^2-k^2}}\tan^{-1}\frac{dx-by}{ay-cx} = 0.$$

(25) A particle moving under gravity in a medium, the retardation due to whose resistance $= \frac{v^2}{a}$, slides in a vertical plane down the curve

$$\frac{y}{b} = e^{\frac{s}{a}} - 1 - \frac{s}{a},$$

where s is the length of the curve measured from the lowest point, y the ordinate of the extremity of this arc referred to a vertical axis, and a a constant; shew that the time of reaching the lowest point is independent of the height from which it starts.

(26) A particle of mass m falls down a smooth cycloid whose axis is vertical and vertex upwards, in a medium whose resistance is $\frac{mv^2}{2c}$, and the distance of the starting point from the vertex is c; prove that the time to the cusp is $\sqrt{\frac{8a}{g}\left(\frac{4a}{c}-1\right)}$, $2a$ being the length of the axis.

(27) A particle moves in a resisting medium; state any reasons, arising from the principle of the conservation of momentum, which render it probable that the resistance at any point varies as the density of the medium at that point, and the square of the velocity of the moving particle.

A particle describes in the medium an ellipse under two attractions to the foci varying inversely as the n^{th} power of the distance; find the density of the medium at any point of the path; and shew that if the attractions vary inversely as the distance, being equal at equal distances, the density varies as the acceleration with which it would move in a non-resisting medium, under the same attraction if it were constrained to move in the ellipse.

(28) A particle is suspended so as to oscillate in a cycloid whose vertex is at the lowest point: if it begin to move from a point distant α from the lowest point measured along the curve, and the medium in which it moves produces a small retardation $\frac{v^2}{a}$, prove that before it next comes to rest energy has been dissipated which is $\frac{8\alpha}{3a}$ of its original value.

CHAPTER VIII.

GENERAL THEOREMS.

220. We propose now to prove some of the general theorems connected with the motion of a particle under any forces, and to investigate the forces requisite for the description of given paths in a given manner. Several of these results have already occurred as immediate deductions from the laws of motion; but to maintain the special character of the work we give more formal analytical demonstrations, though these are certainly superfluous.

221. *If a particle be subject to forces, whose resultant is continually at right angles to the direction of motion; the velocity of the particle will be constant.*

Let R be this resultant, λ, μ, ν its direction cosines, then if m be the mass of the particle, the equations of motion are

$$m\frac{d^2x}{dt^2} = \lambda R,$$

$$m\frac{d^2y}{dt^2} = \mu R,$$

$$m\frac{d^2z}{dt^2} = \nu R.$$

Multiplying by $\frac{dx}{dt}$, ..., adding, and observing that

$$\lambda\frac{dx}{ds} + \mu\frac{dy}{ds} + \nu\frac{dz}{ds} = 0,$$

since the force R is at right angles to the element of the path,

we have $$\frac{1}{2}\frac{d}{dt}(v^2) = \frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2} = 0;$$

therefore $$v = \text{const.}$$

Or, we might at once have resolved along the arc; this would have given

$$\frac{d^2s}{dt^2}=0;$$

whose integral is

$$\frac{ds}{dt}=v=\text{const.}$$

The value of R is evidently $m\dfrac{v^2}{\rho}$; and therefore R varies inversely as the radius of absolute curvature of the path. It is clear that its direction lies in the osculating plane, for there is no acceleration perpendicular to that plane.

Ex. *A particle projected in a plane is under a constant force* R *in that plane continually perpendicular to the direction of motion; to find the path described.*

Here $R=\dfrac{v^2}{\rho}$; and therefore ρ is constant, or the path is a circle.

222. *If* X, Y, Z *be the rectangular components of a force or forces such as occur in nature, i.e. tending to fixed centres and being functions of the distances from these centres,*

$$\mathrm{X}d\mathrm{x}+\mathrm{Y}d\mathrm{y}+\mathrm{Z}d\mathrm{z}=-d\mathrm{V},$$

i.e. is a complete differential. Compare § 78.

Let the points a_1, b_1, c_1; a_2, b_2, c_2; &c. be the positions of the centres of force; x, y, z the co-ordinates of the attracted particle; then, if $r_1, r_2, \ldots$ be its distances from the centres, $\phi_1'(D)$, $\phi_2'(D)$, &c. the laws of attraction to those centres, we have

$$X=\frac{a_1-x}{r_1}\phi_1'(r_1)+\frac{a_2-x}{r_2}\phi_2'(r_2)+\ldots\ldots$$

$$=\Sigma\frac{a-x}{r}\phi'(r).$$

But $$r = \sqrt{\{(a-x)^2 + (b-y)^2 + (c-z)^2\}};$$

which gives $\left(\frac{dr}{dx}\right) = -\frac{a-x}{r}$, &c., for the values of the partial differential coefficients of r.

Hence

$$X = -\Sigma\phi'(r)\left(\frac{dr}{dx}\right),$$

$$Y = -\Sigma\phi'(r)\left(\frac{dr}{dy}\right),$$

$$Z = -\Sigma\phi'(r)\left(\frac{dr}{dz}\right).$$

These give

$$Xdx + Ydy + Zdz$$

$$= -\Sigma\phi'(r)\left\{\left(\frac{dr}{dx}\right)dx + \left(\frac{dr}{dy}\right)dy + \left(\frac{dr}{dz}\right)dz\right\}$$

$$= -\Sigma\phi'(r)\,dr = -dV \quad\text{.............................(1)},$$

since every term of the sum is a complete differential. From § 78 it is obvious that V is the potential at x, y, z.

223. *Under any forces such as occur in nature the increment of the square of the velocity of a particle in passing from one point to another is independent of the path pursued, and depends only on the initial and final positions. This is true even if the particle be forced to move in any particular path by a constraint continually perpendicular to its direction of motion, such as frictionless constraint.*

If we choose tangential resolution, the constraint has no component in that direction, and the equation of motion is

$$\frac{d^2s}{dt^2} = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds};$$

which becomes by (1)

$$\frac{ds}{dt}\frac{d^2s}{dt^2} = -\Sigma\phi'(r)\frac{dr}{dt} = -\frac{dV}{dt}.$$

Therefore $$\tfrac{1}{2}v^2 = C - \Sigma\phi(r) = C - V.$$

Hence, if U be the velocity at a point whose distances from the centres are $R_1, R_2, \ldots\ldots$, and where $V = V_1$,

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}U^2 = \Sigma\phi(R) - \Sigma\phi(r) = V_1 - V,$$

or $$\tfrac{1}{2}v^2 + V = \tfrac{1}{2}U^2 + V_1,$$

a result which involves only the co-ordinates of the initial and final positions. See, again, § 78.

224. Hence if from any point of the surface

$$V = \Sigma\phi(r) = A,$$

a particle be projected with a given velocity in any direction; its velocity when it meets the surface

$$V = \Sigma\phi(r) = B,$$

will be the same, in whatever point it meet that surface; A and B being any constants.

Now on account of equation (1),

$$V = \Sigma\phi(r) = \text{constant}$$

is the equation of a surface on which if smooth a particle will rest in any position under the action of the given forces.

Hence a particle leaving any point of a surface of equilibrium with a given velocity, will have on reaching any other surface of equilibrium a velocity independent of the path pursued or the point reached. This is evident from § 78 if we notice that a surface of equilibrium is an *Equipotential Surface.*

225. *To find the condition to which the applied forces must be subject when the kinetic energy of a particle depends upon its position only.* This is merely the converse of § 223.

Here we have

$$\frac{1}{2}v^2 = \phi(x, y, z),$$

and, therefore,

$$vdv = \left(\frac{d\phi}{dx}\right)dx + \left(\frac{d\phi}{dy}\right)dy + \left(\frac{d\phi}{dz}\right)dz.$$

But, in all cases of motion,

$$vdv = Xdx + Ydy + Zdz.$$

Hence, in this case we must have

$$X = \left(\frac{d\phi}{dx}\right), \quad Y = \left(\frac{d\phi}{dy}\right), \quad Z = \left(\frac{d\phi}{dz}\right);$$

that is,

$$Xdx + Ydy + Zdz$$

must be the differential of a function of three independent variables.

If the seat of the force be in a definite fixed point, which may be taken as origin, the velocity can evidently depend only on the *distance* from that point, not on the *direction* of the distance; hence, if

$$r = \sqrt{x^2 + y^2 + z^2},$$

we have

$$\frac{1}{2}v^2 = \phi(r).$$

The above process gives, in this case,

$$vdv = Xdx + Ydy + Zdz = d\phi(r)$$

$$= \phi'(r)\left(\frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz\right),$$

or

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z},$$

which shew that *the force is in the direction of* r.

From this again it evidently follows that *its magnitude must be a function of* r.

226. The proposition of § 223 contains the Principle of the Conservation of Energy for the case of a single particle.

From this principle it follows that if several particles moving under the influence of the same centre of attraction have equal velocities at any particular distance from their centre; their velocities will always be equal at equal distances from that centre.

Now we have seen (§ 151) that the axis major, $2a$, of an elliptic orbit about a centre of attraction in the focus is independent of the direction of projection. Hence, by considering the particular case of a very narrow ellipse, we see that the velocity at any point is due to a fall, from rest at a distance $2a$, to that point; and that, therefore, in any elliptic orbit about a focus the velocity at any point is that due to a fall to the point, through a distance equal to the distance from the other focus.

227. *If the forces acting on a particle, and the square of its velocity, be increased at any instant in the same ratio, the path will not be altered.*

For the tangent, and the osculating plane, which contains the tangent and the resultant force, are evidently not altered. And the curvature, being

$$\frac{\text{Normal Component of Forces}}{\text{Square of velocity}},$$

has its numerator and denominator increased in the same ratio. And the square of the velocity at the end of any arc is increased in the same ratio as that at the beginning. Hence each successive elementary arc of the path remains unchanged.

228. *If a number of separate particles whose masses are* m_1, m_2, *&c. subjected to forces* f_1, f_2, *&c. respectively, and successively projected from the same point in the same direc-*

tion with velocities $\overline{\mathrm{v}}_1$, $\overline{\mathrm{v}}_2$, *&c. all describe one path; the same path will also be described by a particle of mass* M *projected with velocity* $\overline{\mathrm{U}}$ *from the same point in the same direction, and acted on at once by the same forces* f_1, f_2, *&c. provided*

$$\mathrm{M}\overline{\mathrm{U}}^2 = \Sigma(\mathrm{m}\overline{\mathrm{v}}^2).$$

Suppose that, in addition to the forces f_1, f_2, &c., a force R continually acting in a direction at right angles to that of M's motion be required to cause it to move in the given path; i.e. suppose M to be constrained by a smooth tube to move in the required path; the equations of motion are

$$M\frac{d^2x}{dt^2} = \Sigma mX + R\lambda \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

with similar equations in y and z,

where λ, μ, ν are the direction cosines of R, and X, Y, Z the resolved parts of f.

Multiplying by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ in order, and adding, we eliminate R and have

$$\frac{1}{2}Md(U^2) = \Sigma mXdx + \Sigma mYdy + \Sigma mZdz.$$

But for the separate particles m_1, m_2, &c. we have

$$\frac{1}{2}m_1d(v_1^2) = m_1X_1dx + m_1Y_1dy + m_1Z_1dz, \text{ \&c.};$$

therefore, the path being the same for all,

$$\frac{1}{2}\Sigma\{md(v^2)\} = \Sigma mXdx + \Sigma mYdy + \Sigma mZdz.$$

Hence $\Sigma\{md(v^2)\} = Md(U^2)$,

or $$\Sigma(mv^2) = MU^2 + C.$$

But, by hypothesis, $\Sigma m\bar{v}^2 = M\overline{U}^2$,

therefore $$C = 0.$$

[Instead of this analysis, it is sufficient (by § 78) to notice that the work done on M is the sum of that done on m_1, m_2, &c. Hence the *increase* of kinetic energy must be the same; and if, at starting, the kinetic energy of M be the sum of those of m_1, m_2, &c. it will remain so throughout the motion.]

Hence the kinetic energy of M will be at each point of the orbit equal to the sum of the kinetic energies of m_1, m_2, &c., at that point. To find R, notice that in general the pressure on a constraining curve depends upon two things, the resolved parts of the impressed forces, and the pressure due to the velocity. Now the latter part is as the kinetic energy, therefore in the case of M it is the sum of the corresponding forces in the case of m_1, m_2, &c. Also the same may be said of the resolved parts of the impressed forces. But in the case of each particle, these partial pressures destroyed each other, since the curve was described freely, hence their sums will destroy each other, or the curve will be freely described by M.

229. *If at any instant the velocity of a particle, moving under a conservative system of forces,* § 77, *be reversed, the particle will describe its former path in the reverse direction.*

Suppose a smooth tube, in the form of the original path, requisite to constrain the particle to move backwards along it. The velocity will be, at each point, of the same magnitude as before; the resultant acceleration, and the curvature of the path, will also be alike; hence the normal component of the force will produce the requisite curvature of the path, and there will be no pressure on the constraining tube. The tube is, therefore, not required. Whence the proposition.

230. LEAST, OR STATIONARY, ACTION. If v be the velocity of a particle whose mass is m, and if s be the arc of the path described, the value of the quantity

$$A = m \int v ds$$

(taken between proper limits) is called the *Action* of the particle.

If a particle move freely, or on a smooth surface, (under forces such as occur in nature,) between any two points, the value of the integral m $\int$ vds *for the whole actual path is generally less than it would be if the particle were constrained to pass from one point to the other by a different path.* This, combined with the above definition, is for a single particle the *Principle of Least Action;* of which in an elementary work like the present we can give only a very imperfect sketch. For further information see Thomson and Tait's *Natural Philosophy*, § 318.

231. The proposition to be proved is that, δ being the symbol of the Calculus of Variations, and the mass of the particle being for simplicity taken as unity,

$$\delta A = \delta \int v ds = 0.$$

Now $\delta \int v ds = \int \delta (v ds) = \int (v \delta ds + ds \, \delta v)$

$$= \int (v \delta ds + v dt \delta v), \text{ since } v = \frac{ds}{dt}.$$

But generally,

$$\frac{1}{2} v^2 = \int (X dx + Y dy + Z dz) = \psi (x, y, z),$$

the constraint, if any, having disappeared;

hence $\qquad v \delta v = X \delta x + Y \delta y + Z \delta z.$

But $\qquad X = \dfrac{d^2 x}{dt^2} - R\lambda$, &c.

Hence

$$v \delta v = \left(\frac{d^2 x}{dt^2} \delta x + \frac{d^2 y}{dt^2} \delta y + \frac{d^2 z}{dt^2} \delta z \right) - R (\lambda \delta x + \mu \delta y + \nu \delta z).$$

Now if the particle remain on the surface whose equation is $F = 0$,

$$\lambda \delta x + \mu \delta y + \nu \delta z = k \delta F = 0,$$

and if it leave it $R = 0$, so in either case the latter term on the right vanishes.

Also $$ds^2 = dx^2 + dy^2 + dz^2;$$

which gives $$ds\delta ds = dx\delta dx + dy\delta dy + dz\delta dz,$$

or $$v\delta ds = \frac{dx}{dt}\delta dx + \frac{dy}{dt}\delta dy + \frac{dz}{dt}\delta dz$$

$$= \frac{dx}{dt}d\delta x + \frac{dy}{dt}d\delta y + \frac{dz}{dt}d\delta z,$$

since the order of d and δ is immaterial.

Hence

$$\delta A = \delta \int v ds = \int \left\{ \frac{dx}{dt}d\delta x + \frac{dy}{dt}d\delta y + \frac{dz}{dt}d\delta z \right.$$

$$\left. + \delta x d\left(\frac{dx}{dt}\right) + \delta y d\left(\frac{dy}{dt}\right) + \delta z d\left(\frac{dz}{dt}\right) \right\}$$

$$= \left[\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right],$$

taken between proper limits. Now at both limits

$$\delta x = 0, \quad \delta y = 0, \quad \delta z = 0;$$

hence we have $$\delta A = 0.$$

232. It is commonly said that as, in general, it is impossible to suppose the Action a maximum, this result shews that it is a minimum. The true interpretation of the expression, $\delta A = 0$, is that the unconstrained path of the particle is such, that a small deviation from it will produce an infinitely smaller change in the value of A. Hence Hamilton has suggested the more appropriate title *Stationary Action.*

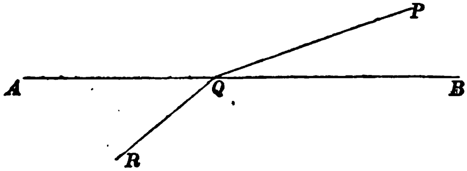
233. If no forces act on the particle except the constraint of the surface, we have v constant, and the above equation shews that in this case the length of the path is generally a minimum.

A particle therefore, projected along a surface and subject to no forces, will trace out between any two points in its path the shortest line on the surface.

It may happen, in the case of a sphere for instance, that the particle will not necessarily trace out the shortest line on the surface between the two points; but we cannot here enter into the details necessary to the full elucidation of such cases.

234. We may apply this principle directly to form the equations of motion in any particular case, or to find the actual path under the action of any forces.

Ex. I. *Let us take again the case of the refraction of light in the corpuscular theory* (§ 130), *as illustrating the general principle of Least Action in the case of a particle.*



The velocity in the upper medium is supposed to be u, that in the lower v, AB being an equipotential surface.

In this case the expression for the Action becomes simply

$$uPQ + vQR,$$

if PQR be the path of the particle, the mass being unity.

By making this quantity a minimum, as depending on the position of Q, P and R being given points; it is easy to shew that Q must lie in the plane through P and R perpendicular to the surface AB, and also that the resolved parts of the velocities in the upper and lower medium parallel to the tangent plane to AB at Q must be equal; and therefore the impulse applied to the corpuscle at Q is perpendicular to AB, while the sines of the angles which PQ and QR make with the perpendicular to AB are inversely as the velocities in the two media.

(If we had made the *Time* from P to R a minimum, we should have obtained the law of refraction on the undulatory theory.)

235. Ex. II. *To find the equation of the path described by a particle about a centre of attraction.*

Let P be the central attraction at distance r, then

$$\frac{1}{2}v^2 = C - \int P dr,$$

$$= \frac{1}{2}\{\phi(r)\}^2, \quad \text{suppose, ...(1)}$$

which gives

$$\int v ds = \int \phi(r) ds.$$

Hence

$$\begin{aligned}
0 &= \delta\int \phi(r)\, ds \\
&= \int\{\phi'(r)\,\delta r\, ds + \phi(r) d\delta s\} \\
&= \int\Big\{\frac{\phi'(r)}{r}(x\delta x + y\delta y + z\delta z)\, ds \\
&\qquad\qquad + \phi(r)\left(\frac{dx}{ds}d\delta x + \frac{dy}{ds}d\delta y + \frac{dz}{ds}d\delta z\right)\Big\} \\
&= \left[\phi(r)\left(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\right)\right] \\
&\quad + \int\Big[\frac{\phi'(r)}{r}(x\delta x + y\delta y + z\delta z)\, ds \\
&\qquad\qquad - \delta x d\left\{\phi(r)\frac{dx}{ds}\right\} - \delta y d\left\{\phi(r)\frac{dy}{ds}\right\} - \delta z d\left\{\phi(r)\frac{dz}{ds}\right\}\Big].
\end{aligned}$$

The integrated part refers only to the limits, and must therefore vanish independently of the integral. That the integral may be identically zero, we must have

$$\frac{x\phi'(r)}{r} - \frac{d}{ds}\left\{\phi(r)\frac{dx}{ds}\right\} = 0,$$

with similar equations in y and z. These may be written

$$\left.\begin{aligned}
\phi'(r)\left(\frac{x}{r} - \frac{dr}{ds}\frac{dx}{ds}\right) - \phi(r)\frac{d^2x}{ds^2} = 0 \\
\phi'(r)\left(\frac{y}{r} - \frac{dr}{ds}\frac{dy}{ds}\right) - \phi(r)\frac{d^2y}{ds^2} = 0 \\
\phi'(r)\left(\frac{z}{r} - \frac{dr}{ds}\frac{dz}{ds}\right) - \phi(r)\frac{d^2z}{ds^2} = 0
\end{aligned}\right\} \text{.............}(a).$$

Multiplying by any three constants, A, B, C, and adding, we have

$$(Ax + By + Cz)\frac{\phi'(r)}{r}$$
$$-\left(A\frac{dx}{ds} + B\frac{dy}{ds} + C\frac{dz}{ds}\right)\phi'(r)\frac{dr}{ds}$$
$$-\left(A\frac{d^2x}{ds^2} + B\frac{d^2y}{ds^2} + C\frac{d^2z}{ds^2}\right)\phi(r) = 0;$$

which is obviously satisfied by

$$Ax + By + Cz = 0.$$

This equation shews that the orbit is in a plane passing through the centre of attraction. Let xy be this plane, then we may confine ourselves to the first two of equations (a).

Multiplying the second by x and the first by y and subtracting, we obtain

$$\phi'(r)\frac{dr}{ds}\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right) + \phi(r)\left(x\frac{d^2y}{ds^2} - y\frac{d^2x}{ds^2}\right) = 0.$$

This is immediately integrable, and gives

$$\phi(r)\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right) = \text{constant}.$$

Since $\phi(r) = v$, we see by § 24, that this is in polar co-ordinates

$$vp = r^2\frac{d\theta}{dt} = h \text{.........................} (b),$$

which is the equation for the equable description of areas.

Finally, multiplying these two first equations of group (a) by x and y respectively and adding, we have

$$r\phi'(r)\left\{1 - \left(\frac{dr}{ds}\right)^2\right\} - \phi(r)\left(x\frac{d^2x}{ds^2} + y\frac{d^2y}{ds^2}\right) = 0 \text{.......} (c).$$

But, since

$$r\frac{dr}{ds} = x\frac{dx}{ds} + y\frac{dy}{ds},$$

we have by differentiation

$$x\frac{d^2x}{ds^2}+y\frac{d^2y}{ds^2}=r\frac{d^2r}{ds^2}+\left(\frac{dr}{ds}\right)^2-1.$$

Substituting in (c), and changing the independent variable from s to θ by means of the equation

$$ds^2=dr^2+r^2d\theta^2,$$

we have

$$\phi'(r)\,r\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}-\phi(r)\left\{r\frac{d^2r}{d\theta^2}-2\left(\frac{dr}{d\theta}\right)^2-r^2\right\}=0.$$

Putting $\frac{1}{u}$ for r, this becomes

$$\frac{d^2u}{d\theta^2}+u=-\frac{\phi'(r)}{\phi(r)}\left\{1+\frac{1}{u^2}\left(\frac{du}{d\theta}\right)^2\right\}\ \ldots\ldots\ldots\ldots(d).$$

But, by (b) as developed in § 142,

$$v^2=\{\phi(r)\}^2=h^2\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}.$$

Also $$\phi(r)\,\phi'(r)=-P,\qquad\text{by (1).}$$

Thus (d) becomes

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2},\qquad\text{as in § 135.}$$

236. We might have treated these equations (§ 235 (a)) somewhat differently thus

$$\phi(r)=v=\frac{ds}{dt}.$$

Hence $$\phi(r)\frac{dx}{ds}=\frac{dx}{dt},\ \&c.;$$

and we have the equations

$$\frac{x\phi'(r)}{r}-\frac{d}{ds}\left(\frac{dx}{dt}\right)=0,\ \&c.\ \&c.,$$

which give, at once,

$$\frac{d\left(\frac{dx}{dt}\right)}{x}=\frac{d\left(\frac{dy}{dt}\right)}{y}=\frac{d\left(\frac{dz}{dt}\right)}{z},$$

containing the theorems of constant plane and equable description of areas; and since

$$\phi'(r)\frac{ds}{dt}=\phi(r)\,\phi'(r)=-P,$$

$$-\frac{x}{r}P-\frac{d^2x}{dt^2}=0,\ \&c.,$$

the ordinary equations in three rectangular directions.

237. We might have simplified the work by using polar co-ordinates immediately after having proved that the orbit is plane. For we have

$$A=\int\phi(r)\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta,\ \text{a minimum,}$$

and therefore the calculus of variations gives (by the formula $V=Pp+C$)

$$\phi(r)\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}=\phi(r)\frac{\left(\frac{dr}{d\theta}\right)^2}{\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}}+C,$$

or reducing, and putting h for C,

$$r^2\phi(r)=h\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\ \ldots\ldots\ldots\ldots\ldots\ldots(e),$$

or $$r^2\frac{ds}{dt}=h\frac{ds}{d\theta};$$

whence $$r^2\frac{d\theta}{dt}=h,$$

the equation for the equable description of areas.

Squaring (*e*) and attending to (1), we have

$$2\frac{r^4}{h^2}(C - \int P dr) = r^2 + \left(\frac{dr}{d\theta}\right)^2,$$

or, putting $$r = \frac{1}{u},$$

$$2\frac{1}{h^2}\left(C + \int \frac{P}{u^2}\frac{du}{d\theta}\,d\theta\right) = u^2 + \left(\frac{du}{d\theta}\right)^2,$$

and differentiating and dividing by $2\dfrac{du}{d\theta}$,

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2},$$

the general equation of central orbits.

238. VARYING ACTION. If, in § 231, we assume

$$\tfrac{1}{2}v^2 = \int (Xdx + Ydy + Zdz) + H = H - V,$$

(with the notation of § 78) it is evident that H will depend on the initial velocity. Supposing that this and the initial and final co-ordinates vary; then, in addition to the already considered variation of the form of the path between its extremities, upon which the unintegrated part of the value of δA depends, we shall have in δA terms depending on the variations of initial and final positions and of initial velocity.

The additional term in $v\delta v$ is δH, and its integral $t\delta H$ is at once obtained. Hence in this more general variation of the conditions we have in the value of δA the following additional terms, depending on the limits only, and therefore to be treated by themselves,

$$\delta A = \left[\frac{dx}{dt}\,\delta x + \frac{dy}{dt}\,\delta y + \frac{dz}{dt}\,\delta z\right]$$
$$- \left[\left(\frac{dx}{dt}\right)_0 \delta x_0 + \left(\frac{dy}{dt}\right)_0 \delta y_0 + \left(\frac{dz}{dt}\right)_0 \delta z_0\right] + t\delta H.$$

Hence, if A could be found in terms of x, y, z, x_0, y_0, z_0, and H, we should have at once the first and second integrals of the equations of motion in the form

$$\left(\frac{dA}{dx}\right) = \frac{dx}{dt}, \quad \left(\frac{dA}{dx}\right)_0 = -\left(\frac{dx}{dt}\right)_0,$$
$$\&c. \qquad \&c.,$$

with the farther condition

$$\left(\frac{dA}{dH}\right) = t.$$

239. A is, of course, a function of x, y, z, x_0, y_0, z_0, and H, and we see by the equations above that it must satisfy the partial differential equations

$$\left(\frac{dA}{dx}\right)^2 + \left(\frac{dA}{dy}\right)^2 + \left(\frac{dA}{dz}\right)^2 = v^2 = 2(H - V) \ldots\ldots (1),$$

and
$$\left(\frac{dA}{dx_0}\right)^2 + \left(\frac{dA}{dy_0}\right)^2 + \left(\frac{dA}{dz_0}\right)^2 = 2(H - V_0) \ldots\ldots\ldots\ldots (2).$$

240. The whole circumstances of the motion are thus dependent on the function A, called by Hamilton the *Characteristic Function*. The above is a brief sketch of the foundation of his theory of *Varying Action*, so far as it relates to the motion of a single free particle. The determination of the function A is troublesome, even in very simple cases of motion; but the fact that such a mode of representation is possible is extremely remarkable.

241. More generally, omitting all reference to the initial point, and the equation § 239 (2) which belongs to it, let us consider A simply as a function of x, y, z. Then

Any function, A, *which satisfies*

$$\left(\frac{dA}{dx}\right)^2 + \left(\frac{dA}{dy}\right)^2 + \left(\frac{dA}{dz}\right)^2 = v^2 = 2(H - V)$$

possesses the property that

$$\frac{dA}{dx}, \quad \frac{dA}{dy}, \quad \frac{dA}{dz}$$

represent the rectangular components of the velocity of a particle in a motion possible under the forces whose potential is V.

For, by partial differentiation of § 239, (1), we have

$$\frac{d^2x}{dt^2} = X = -\frac{dV}{dx} = \frac{dA}{dx}\frac{d^2A}{dx^2} + \frac{dA}{dy}\frac{d^2A}{dxdy} + \frac{dA}{dz}\frac{d^2A}{dxdz}.$$

But $$\frac{d}{dt}\left(\frac{dA}{dx}\right) = \frac{dx}{dt}\frac{d^2A}{dx^2} + \frac{dy}{dt}\frac{d^2A}{dxdy} + \frac{dz}{dt}\frac{d^2A}{dxdz}.$$

Comparing, we see that

$$\frac{dx}{dt} = \frac{dA}{dx}, \quad \frac{dy}{dt} = \frac{dA}{dy}, \quad \frac{dz}{dt} = \frac{dA}{dz},$$

satisfy this and the other two similar pairs of equations.

242. Also, if α, β be constants, which, along with H, are involved in a complete integral of the above partial differential equation, *the corresponding path, and the time of its description, are given by*

$$\left(\frac{dA}{d\alpha}\right) = \alpha_1, \quad \left(\frac{dA}{d\beta}\right) = \beta_1, \quad \left(\frac{dA}{dH}\right) = t + \epsilon,$$

where α_1, β_1, ϵ *are three additional constants.*

For these equations give, by differentiation,

$$\left.\begin{aligned} \frac{d^2A}{dxd\alpha}\frac{dx}{dt} + \frac{d^2A}{dyd\alpha}\frac{dy}{dt} + \frac{d^2A}{dzd\alpha}\frac{dz}{dt} &= 0 \\ \frac{d^2A}{dxd\beta}\frac{dx}{dt} + \frac{d^2A}{dyd\beta}\frac{dy}{dt} + \frac{d^2A}{dzd\beta}\frac{dz}{dt} &= 0 \\ \frac{d^2A}{dxdH}\frac{dx}{dt} + \frac{d^2A}{dydH}\frac{dy}{dt} + \frac{d^2A}{dzdH}\frac{dz}{dt} &= 1 \end{aligned}\right\} \text{...........}(a).$$

But, differentiating § 239, (1), we get

$$\left.\begin{aligned} \frac{d^2A}{d\alpha dx}\frac{dA}{dx} + \frac{d^2A}{d\alpha dy}\frac{dA}{dy} + \frac{d^2A}{d\alpha dz}\frac{dA}{dz} &= 0 \\ \frac{d^2A}{d\beta dx}\frac{dA}{dx} + \frac{d^2A}{d\beta dy}\frac{dA}{dy} + \frac{d^2A}{d\beta dz}\frac{dA}{dz} &= 0 \\ \frac{d^2A}{dHdx}\frac{dA}{dx} + \frac{d^2A}{dHdy}\frac{dA}{dy} + \frac{d^2A}{dHdz}\frac{dA}{dz} &= 1 \end{aligned}\right\} \text{........}(b).$$

The values of $\frac{dx}{dt}$, &c. in (*a*) are evidently equal respectively to those of $\left(\frac{dA}{dx}\right)$, &c. in (*b*). Hence the proposition.

243. *Equiactional surfaces*, i.e. those whose common equation is

$$A = \text{const.} = C,$$

are cut at right angles by the trajectories.

For the direction-cosines of the normal are obviously proportional to $\left(\frac{dA}{dx}\right)$, $\left(\frac{dA}{dy}\right)$, $\left(\frac{dA}{dz}\right)$, that is to $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$.

Thus the determination of equiactional surfaces is resolved into the problem of finding the orthogonal trajectories of a set of given curves in space, whenever the conditions of the motion are given. We cannot, in the present work, spare space for much detail on this very curious subject, and therefore give but one other singular property of these surfaces before applying the principle of Varying Action to an important problem.

Let ϖ be the normal distance at any point between the consecutive surfaces

$$A = C, \text{ and } A = C + \delta C.$$

We have evidently

$$\left(\frac{dA}{dx}\right)\delta x + \left(\frac{dA}{dy}\right)\delta y + \left(\frac{dA}{dz}\right)\delta z = \delta C,$$

or

$$\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z = \delta C,$$

where δx, δy, δz are the relative co-ordinates of any two contiguous points on the two surfaces. If ρ be the length of the line joining these points, θ its inclination to the normal (i.e. the line of motion), this may evidently be written

$$v\rho \cos\theta = v\varpi = \delta C,$$

since $\rho \cos\theta$ is the normal distance between the surfaces.

Thus, *the distance between consecutive equiactional surfaces is, at any point, inversely as the velocity in the corresponding path.*

This may be seen at once as follows; the element of the action is $v\delta s$ (where δs, being an element of the path, is the normal distance between the surfaces) and must therefore be equal to δC.

244. *To deduce, from the principle of Varying Action, the form and mode of description of a planet's orbit.*

In this case it is obvious that $-\dfrac{dV}{dr}$ represents the attraction of gravity $\left(-\dfrac{\mu}{r^2}\right)$. Hence the right-hand member of § 239 (1) may be written $2\left(H+\dfrac{\mu}{r}\right)$.

Let us take the plane of xy as that of the orbit, then the equation § 239 (1) becomes

$$v^2=\left(\frac{dA}{dx}\right)^2+\left(\frac{dA}{dy}\right)^2=2\left(H+\frac{\mu}{r}\right) \quad\text{..........(1).}$$

It is not difficult to obtain a satisfactory solution of this equation; but the operation is very much simplified by the use of polar co-ordinates. With this change, (1) becomes

$$\left(\frac{dA}{dr}\right)^2+\frac{1}{r^2}\left(\frac{dA}{d\theta}\right)^2=2\left(H+\frac{\mu}{r}\right) \quad\text{............(2),}$$

which is obviously satisfied by

$$\left.\begin{aligned}\left(\frac{dA}{d\theta}\right)&=\text{constant}=\alpha\\ \left(\frac{dA}{dr}\right)^2&=2\left(H+\frac{\mu}{r}\right)-\frac{\alpha^2}{r^2}\end{aligned}\right\} \quad\text{...............(3).}$$

Hence

$$A=\alpha\theta+\int dr\sqrt{2\left(H+\frac{\mu}{r}\right)-\frac{\alpha^2}{r^2}} \quad\text{...........(4).}$$

The final integrals are therefore, by § 242,

$$\left(\frac{dA}{d\alpha}\right)=\alpha_1=\theta-\alpha\int\frac{dr}{r^2\sqrt{2\left(H+\frac{\mu}{r}\right)-\frac{\alpha^2}{r^2}}}\text{..........(5)},$$

and

$$\left(\frac{dA}{dH}\right)=t+\epsilon=\int\frac{dr}{\sqrt{2\left(H+\frac{\mu}{r}\right)-\frac{\alpha^2}{r^2}}}\text{............(6)}.$$

These equations contain the complete solution of the problem, for they involve four constants, α_1, α, H, ϵ. (5) gives the equation of the orbit, and (6) the time in terms of the radius vector.

245. To complete the investigation, let us assume

$$\frac{\mu}{\alpha^2}=\frac{1}{l},$$

$$\frac{2H}{\alpha^2}=\frac{e^2-1}{l^2},$$

where l and e are two new arbitrary constants introduced in place of α and H. With these (5) becomes

$$\alpha_1=\theta-\int\frac{dr}{r^2\sqrt{\frac{e^2-1}{l^2}+\frac{2}{lr}-\frac{1}{r^2}}}$$

$$=\theta-\int\frac{dr}{r^2\sqrt{\frac{e^2}{l^2}-\left(\frac{1}{r}-\frac{1}{l}\right)^2}}$$

$$=\theta-\cos^{-1}\frac{\frac{1}{r}-\frac{1}{l}}{\frac{e}{l}},$$

or

$$r=\frac{l}{1+e\cos(\theta-\alpha_1)},$$

the general polar equation of conic sections referred to the focus.

Also, by differentiating (5) with respect to r, we have

$$\frac{\alpha dr}{r^2\sqrt{2\left(H+\frac{\mu}{r}\right)-\frac{\alpha^2}{r^2}}}=d\theta,$$

from which, by (6), we immediately obtain

$$t+\epsilon=\frac{1}{\alpha}\int r^2 d\theta=\frac{1}{\sqrt{\mu l}}\int r^2 d\theta.$$

This involves, again, the equation of equable description of areas.

246. To illustrate the subject farther, we will deduce others of the ordinary results of Chaps. V. and VI. from these formulæ. Thus, let θ_0, r_0 denote the polar co-ordinates of any fixed point in the path, from which the action is to be reckoned. We have, by (4),

$$A=\alpha(\theta-\theta_0)+\int_{r_0}^{r} dr\sqrt{2\left(\frac{\mu}{r}+H\right)-\frac{\alpha^2}{r^2}}$$

$$=\int_{r_0}^{r}\frac{2\left(\frac{\mu}{r}+H\right)dr}{\sqrt{2\left(\frac{\mu}{r}+H\right)-\frac{\alpha^2}{r^2}}}\text{..........................(7)},$$

because, by (5),

$$\theta-\theta_0=\int_{r_0}^{r}\frac{\alpha dr}{r^2\sqrt{2\left(\frac{\mu}{r}+H\right)-\frac{\alpha^2}{r^2}}}.$$

To integrate (7), remark that (§ 149) $\frac{v^2}{2}<\frac{\mu}{r}$ in an elliptic orbit, and that thus H is negative by § 244 (1).

Put
$$\frac{\mu}{H} = -2a$$
$$\frac{a^2}{\mu a} = 1 - e^2,$$
and
$$r = a(1 - e\cos\phi),$$
and (7) becomes, after substitution,
$$A = \sqrt{\mu a}\int_{\phi_0}^{\phi}(1 + e\cos\phi)\,d\phi,$$
which is immediately integrable.

It is obvious from § 160 that ϕ represents the excentric anomaly. If we measure it from the perihelion we have evidently
$$A = \sqrt{\mu a}\,(\phi + e\sin\phi).$$

247. By (6) we have
$$t = \int_{r_0}^{r}\frac{dr}{\sqrt{2\left(\frac{\mu}{r} + H\right) - \frac{a^2}{r^2}}}.$$

By employing the same substitutions as in last section, ϕ being measured from perihelion, it is easy to bring this expression into the form
$$t = \sqrt{\frac{a^3}{\mu}}\int_0^{\phi}(1 - e\cos\phi)\,d\phi$$
$$= \sqrt{\frac{a^3}{\mu}}\,[\phi - e\sin\phi],$$
the formula of § 160.

248. By the process of § 160 we see that while
$$\phi - e\sin\phi$$
is proportional to the area described about the centre of attraction, and therefore proportional to the time;
$$\phi + e\sin\phi$$
is proportional to the area described about the other focus, and is, by § 246, proportional to the Action. Thus in a

planet's elliptic orbit *the time is measured by the area described about one focus, and the Action by that about the other.*

An easy verification of this curious result is as follows. With the usual notation we have

$$dA = vds$$
$$= \frac{h}{p}ds,$$

by the result of § (141). But in the ellipse or hyperbola, p being the perpendicular from the second focus,

$$pp' = \pm b^2.$$

Hence $$dA = \pm \frac{h}{b^2}p'ds,$$

which expresses the result sought. (*Proc. R. S. E.*, March, 1865.)

It is easy to extend this to a parabolic orbit, for which, indeed, the theorem is even more simple.

249. It may be useful to give another example of Hamilton's remarkable method. For this purpose we will again briefly consider *Cotes' Spirals.* [See Chap. V., Ex. (9).]

Here the central attraction is inversely as the cube of the distance, and therefore the equation of Action is

$$\left(\frac{dA}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dA}{d\theta}\right)^2 = 2\left(H + \frac{\mu}{r^2}\right).$$

Hence, as in § (244), we have

$$\left(\frac{dA}{d\theta}\right) = \alpha,$$

$$\left(\frac{dA}{dr}\right) = \sqrt{2H + \frac{2\mu - \alpha^2}{r^2}}.$$

From these it is easy to find A, but we leave this as an exercise to the student.

Again,

$$\alpha_1 = \left(\frac{dA}{d\alpha}\right) = \theta - \alpha\int \frac{dr}{r^2\sqrt{2H + \frac{2\mu - \alpha^2}{r^2}}}$$

$$= \theta + \frac{\alpha}{\sqrt{2\mu - \alpha^2}} \log\left\{\frac{\sqrt{2\mu - \alpha^2}}{r} + \sqrt{2H + \frac{2\mu - \alpha^2}{r^2}}\right\}.$$

Substituting exponentials for the logarithm, this takes the form

$$2\frac{\sqrt{2\mu - \alpha^2}}{r} = \epsilon^{-\frac{\sqrt{2\mu - \alpha^2}}{\alpha}(\theta - \alpha_1)} - 2H\epsilon^{\frac{\sqrt{2\mu - \alpha^2}}{\alpha}(\theta - \alpha_1)}.$$

This integration fails for certain special values of, or relations among, the constants, but the reader can have no difficulty in obtaining the requisite changes in these cases, and so reproducing all the varieties of possible orbits given in the Examples to Chap. V.

250. Assuming, for a set of particles, the result of § 231, we may easily obtain the celebrated *equations of motion in generalized co-ordinates* due to Lagrange, as well as the general equations of Varying Action in the form given by Hamilton. The following is an outline of the process for the special case in which the geometrical relations are independent of the time, and in which therefore the conservation of energy holds.

Let the co-ordinates of the particles of such a system be expressed in terms of new co-ordinates $\theta, \phi, \psi, \ldots$ which are independent of one another. Then it is easily shewn (Thomson and Tait's *Nat. Phil.* § 313) that the kinetic energy, T, is a homogeneous quadratic function of $\dot\theta, \dot\phi, \dot\psi, \ldots$ and also a function of $\theta, \phi, \psi, \ldots$

Hence $$\Sigma\dot\theta\left(\frac{dT}{d\dot\theta}\right) = 2T,$$

where the bracket denotes partial differentiation.

But the equation of energy is

$$T + V = H.$$

The variation of the action is

$$\begin{aligned}\delta A &= \delta\int 2T dt \\ &= \delta\int (T+H-V)\,dt \\ &= \int\Sigma\left\{\left(\frac{dT}{d\theta}\right)\delta\theta+\left(\frac{dT}{d\dot\theta}\right)\delta\dot\theta-\left(\frac{dV}{d\theta}\right)\delta\theta\right\}dt+t\delta H \\ &= \Sigma\delta\theta\left(\frac{dT}{d\dot\theta}\right)+t\delta H-\int\Sigma\left\{\frac{d}{dt}\left(\frac{dT}{d\dot\theta}\right)-\left(\frac{dT}{d\theta}\right)+\left(\frac{dV}{d\theta}\right)\right\}\delta\theta dt.\end{aligned}$$

As we have agreed to assume the results of § 231, it is obvious that the unintegrated part of the value of δA must vanish. Hence we have two sets of equations.

1. From the unintegrated part we have Lagrange's Equations, equal in number to the generalized co-ordinates, and of which the following is one:

$$\frac{d}{dt}\left(\frac{dT}{d\dot\theta}\right)-\left(\frac{dT}{d\theta}\right)+\left(\frac{dV}{d\theta}\right)=0.$$

2. From the integrated part the Hamiltonian system

$$\left(\frac{dA}{d\theta}\right)=\left(\frac{dT}{d\dot\theta}\right),\ \&c.$$

along with

$$\left(\frac{dA}{dH}\right)=t.$$

As a verification, differentiate with regard to t the equation

$$\Sigma\theta\left(\frac{dT}{d\dot\theta}\right)=2T=T+H-V,$$

and we have the result

$$\Sigma\dot\theta\left\{\frac{d}{dt}\left(\frac{dT}{d\dot\theta}\right)-\left(\frac{dT}{d\theta}\right)+\left(\frac{dV}{d\theta}\right)\right\}=0,$$

which is obviously consistent with the equations of Lagrange.

251. As an example of Lagrange's equations of motion, consider the case of the small oscillations in the magnetic meridian of two equal bar-magnets each suspended by two equal parallel strings from points in a horizontal line.

Let m be the mass of each magnet, $2a$ the distance between the adjacent poles when the magnets are in equilibrium and demagnetized, l the length of each string, and μ the product of the strengths of the poles.

If x, y be the displacements of the magnets at the time t; then, neglecting the vertical velocities,

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2),$$

$$V = \frac{\mu}{2a + x - y} + \frac{1}{2} m \frac{g}{l} (x^2 + y^2) + \ldots$$

only the contiguous poles of the magnets being supposed to act on one another.

Hence the equations of motion are

$$\frac{d}{dt}(m\dot{x}) = -\frac{\mu}{(2a + x - y)^2} - m\frac{g}{l} x \;\ldots\ldots\ldots\ldots(1),$$

$$\frac{d}{dt}(m\dot{y}) = \frac{\mu}{(2a + x - y)^2} - m\frac{g}{l} y \;\ldots\ldots\ldots\ldots(2).$$

Adding, $$\frac{d}{dt}(\dot{x} + \dot{y}) = -\frac{g}{l}(x + y).$$

Subtracting,

$$m\frac{d}{dt}(\dot{x} - \dot{y}) = -\frac{\mu}{2a^2}\left(1 - \frac{x - y}{a} \ldots\right) - m\frac{g}{l}(x - y).$$

Making x and y constant in (1) and (2) we get their equilibrium values; and measuring x and y from these we get

$$\frac{d^2}{dt^2}(x + y) = -\frac{g}{l}(x + y),$$

$$\frac{d^2}{dt^2}(x - y) = -\left(\frac{g}{l} - \frac{\mu}{2ma^3}\right)(x - y).$$

Thus if $$n^2 = \frac{g}{l},$$

$$n_1^2 = \frac{g}{l} - \frac{\mu}{2ma^3},$$

we have $$x + y = A \cos (nt + B),$$

$$x - y = A_1 \cos (n_1 t + B_1).$$

It depends upon whether the proximate poles of the magnets attract or repel one another, whether n or n_1 is the greater.

If the magnets be swung as one piece at their equilibrium distance from one another, the time of oscillation will be the same as that of either magnet when left to itself, since the magnetic attraction does not vary: this is the character of the first harmonic motion.

Again, if the magnets be swung with equal and opposite motion, the centre of inertia is fixed, and the time of oscillation will be the same as if one of the magnets were held fixed and its magnetic strength doubled; it will therefore be shorter or longer than the first period, according as the poles presented to one another attract or repel; this is the character of the second harmonic motion.

252. If we treat the investigation of § (184), in the way in which Hamilton treated that of § (230), we arrive at a number of curious theorems connected with Brachistochrones; of which a few will be given here from the *Trans. R. S. E.* 1865.

Putting τ for the time in the Brachistochrone, we have

$$\left(\frac{d\tau}{dx}\right) = \frac{1}{v^2}\frac{dx}{dt}, \quad \left(\frac{d\tau}{dx_0}\right) = -\left(\frac{1}{v^2}\frac{dx}{dt}\right)_0$$

....

$$\left(\frac{d\tau}{dH}\right) = -\int \frac{dt}{v^2} = -\int \frac{ds}{v^3},$$

corresponding to the group in § (238).

Hence, just as in § (241) it may be shewn that for any forces, of which V' is the potential, a value of τ from the equation

$$\left(\frac{d\tau}{dx}\right)^2 + \left(\frac{d\tau}{dy}\right)^2 + \left(\frac{d\tau}{dz}\right)^2 = \frac{1}{v^2} = \frac{1}{2(H-V)},$$

is such that its partial differential coefficients represent the components of velocity in a possible brachistochrone, each divided by the square of the whole velocity.

Also if τ contain, besides H, two arbitrary constants, α, β, the equations of the brachistochrone are

$$\left(\frac{d\tau}{d\alpha}\right) = \mathfrak{A},$$

$$\left(\frac{d\tau}{d\beta}\right) = \mathfrak{B}.$$

253. *To find the Brachistochrone when the attraction is central, and proportional to a power of the distance; the velocity being also proportional to a power of the distance, that is, being the velocity from infinity, for an attraction, from the centre for a repulsion.*

Here $$v^2 = 2(H-V) = \frac{\mu}{r^n},$$

and the central attraction at distance r is evidently

$$\frac{dV}{dr} = \frac{n\mu}{2r^{n+1}}.$$

Thus (2) becomes

$$\left(\frac{d\tau}{dx}\right)^2 + \left(\frac{d\tau}{dy}\right)^2 + \left(\frac{d\tau}{dz}\right)^2 = \frac{r^n}{\mu},$$

or, changing to polar co-ordinates,

$$\left(\frac{d\tau}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{d\tau}{d\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{d\tau}{d\phi}\right)^2 = \frac{r^n}{\mu}.$$

It is obvious that we must take

$$\left(\frac{d\tau}{d\phi}\right) = 0,$$

which shews that the path is in a plane passing through the centre of force. The above equation will then be satisfied by

$$\left(\frac{d\tau}{d\theta}\right)=a, \quad \left(\frac{d\tau}{dr}\right)=\sqrt{\frac{r^n}{\mu}-\frac{a^2}{r^2}}.$$

Hence we have

$$\tau=a\theta+\int dr\sqrt{\frac{r^n}{\mu}-\frac{a^2}{r^2}}$$

$$=a\theta+\frac{2a}{n+2}\left\{\sqrt{\frac{r^{n+2}}{\mu a^2}-1}-\cos^{-1}\frac{\sqrt{\mu}a}{r^{\frac{n+2}{2}}}\right\}+C.$$

And the equation of the brachistochrone, which is evidently a plane curve, is

$$\mathfrak{A}=\theta+\frac{2}{n+2}\left\{\sqrt{\frac{r^{n+2}}{\mu a^2}-1}-\cos^{-1}\frac{\sqrt{\mu}a}{r^{\frac{n+2}{2}}}\right\}$$

$$+\frac{2a}{n+2}\left\{-\frac{\frac{r^{n+2}}{\mu a^3}}{\sqrt{\frac{r^{n+2}}{\mu a^2}-1}}+\frac{\sqrt{\mu}}{r^{\frac{n+2}{2}}}\frac{1}{\sqrt{1-\frac{\mu a^2}{r^{n+2}}}}\right\}$$

$$=\theta-\frac{2}{n+2}\cos^{-1}\frac{\sqrt{\mu}a}{r^{\frac{n+2}{2}}}:$$

or

$$r^{\frac{n+2}{2}}=\sqrt{\mu}a\sec\frac{n+2}{2}(\theta-\mathfrak{A}),$$

while the equation of the *free* path is

$$\left(\frac{r}{a}\right)^{\frac{n-2}{2}}=\cos\frac{n-2}{2}(\theta+\beta).$$

The above integration fails in the case of $n=-2$; that is, for a repulsion directly as the distance, the circle of zero velocity being evanescent. But in this case

$$\tau=a\theta+\sqrt{\frac{1}{\mu}-a^2}\log\frac{r}{b},$$

and the equation of the brachistochrone is

$$\mathfrak{A} = \theta - \frac{a}{\sqrt{\frac{1}{\mu} - a^2}} \log \frac{r}{b},$$

the logarithmic spiral. Eliminating r between these equations, we see that the time is proportional to the polar angle.

Since a definite form has been assigned to the expression for the velocity in this problem, it is obvious that H is given, and therefore that there is no $\left(\frac{d\tau}{dH}\right)$.

254. It is easily seen that

$$\tau = C$$

is the equation of an *Isochronous* surface.

Also, since

$$\frac{\left(\frac{d\tau}{dx}\right)}{\frac{dx}{dt}} = \frac{\left(\frac{d\tau}{dy}\right)}{\frac{dy}{dt}} = \frac{\left(\frac{d\tau}{dz}\right)}{\frac{dz}{dt}},$$

the brachistochrone cuts all such surfaces at right angles.

And the normal distance between two consecutive isochronous surfaces is proportional to the velocity in the brachistochrone of which it forms an element. For, of course,

$$\delta s = v\delta\tau.$$

255. Hamilton's equation for the determination of the Characteristic Function (A) in the case of the free motion of a single particle is

$$\left(\frac{dA}{dx}\right)^2 + \left(\frac{dA}{dy}\right)^2 + \left(\frac{dA}{dz}\right)^2 = 2(H - V).$$

The comparison of this with the equation of § 252 suggests a useful transformation. Introducing in that equation a factor θ^2, an undetermined function of x, y, z, we have

$$\left(\theta \frac{d\tau}{dx}\right)^2 + \left(\theta \frac{d\tau}{dy}\right)^2 + \left(\theta \frac{d\tau}{dz}\right)^2 = \frac{\theta^2}{2(H - V)}.$$

If we make

$$\theta = \phi'(\tau)$$

and

$$\frac{\theta^2}{2(H-V)} = 2(H_1 - V_1),$$

it becomes

$$\left(\frac{d\phi(\tau)}{dx}\right)^2 + \left(\frac{d\phi(\tau)}{dy}\right)^2 + \left(\frac{d\phi(\tau)}{dz}\right)^2 = 2(H_1 - V_1).$$

Here it is obvious, that $\phi(\tau)$ is the action in a *free* path coinciding with the brachistochrone, and that $2(H_1 - V_1)$ is the square of the velocity in this path.

Hence the curious result that, *if τ be the time through any arc of a given brachistochrone, the same path will be described freely under forces whose potential is* V_1, *where*

$$2(H_1 - V_1) = \frac{\{\phi'(\tau)\}^2}{2(H-V)},$$

ϕ' being any function whatever, and $\phi(\tau)$ will represent the action in the free path.

256. The simplest supposition we can make is that $\phi'(\tau)$ is constant. In this case the velocity in the free path is inversely proportional to that in the brachistochrone at the same point; and the action in the one is proportional to the time in the other. In fact, as Sir W. Thomson has pointed out, in this case the investigation may be made with extreme simplicity, thus—

In the brachistochrone we have

$$\int \frac{ds}{v} \text{ a minimum.}$$

Putting $\nu = \frac{1}{v}$, and considering ν as the velocity in the same path due to another (easily determinable) potential; we must have

$$\int \nu ds \text{ a minimum.}$$

This is the ordinary condition of *Least Action*, and belongs, therefore, to a free path.

Hence, since the cycloid is the brachistochrone for gravity, and since in it $v^2 = 2gy$, it will be a free path if $v^2 = \dfrac{1}{2gy}$, that is for a system of force where the potential is found from

$$H_1 - V_1 = \frac{1}{4gy}.$$

This gives

$$-\frac{dV_1}{dx} = 0, \qquad -\frac{dV_1}{dy} = -\frac{1}{4gy^2}.$$

In other words, a cycloid may be described freely under a repulsion inversely as the square of the distance from the base; and the velocity at any point will be the reciprocal of that in the same cycloid when it is the common brachistochrone.

This result is easily verified by a direct process.

257. The converse of the proposition in § 255 is also curious. Taking Hamilton's equation, § 239, we have

$$\{\phi'(A)\}^2 \left\{\left(\frac{dA}{dx}\right)^2 + \left(\frac{dA}{dy}\right)^2 + \left(\frac{dA}{dz}\right)^2\right\} = 2(H-V)\{\phi'(A)\}^2.$$

Comparing this with that of § 252, we see that $\tau = \phi(A)$ is the brachistochronic expression for the time in a path which is a free path for potential V, provided that $\phi(A)$ and the potential for the brachistochrone are connected by the equation

$$\frac{1}{2(H_1 - V_1)} = 2(H-V)\{\phi'(A)\}^2.$$

Hence, *if* A *be the action in a given free path, the same path will be a brachistochrone for forces whose potential is* V_1, *determined by the condition just given,* V *being the potential in the free path.*

Thus, the parabola

$$(x - \mathfrak{A})^2 = 4a(y - a)$$

is the free path for $v^2 = 2gy$. And the action is given by

$$\frac{1}{\sqrt{2g}} A = x\sqrt{a} + \frac{2}{3}(y-a)^{\frac{3}{2}}.$$

Hence this parabola is the brachistochrone for

$$2(H_1 - V_1) = \frac{1}{2gy\{\phi'(A)\}^2}.$$

In the simplest case $\phi'(A) = 1$, and we have

$$-\frac{dV_1}{dx} = 0, \quad -\frac{dV_1}{dy} = -\frac{1}{4gy^2}.$$

Hence, by § 256, the parabola is a brachistochrone when a cycloid is the free path.

258. The examples immediately preceding are but particular cases of the following general theorem, which is easily seen to be involved in the results of §§ 255, 257. *If we have two curves* P *and* Q, *of which* P *is a free path, and* Q *a brachistochrone, for a given conservative system of forces;* P *will be a brachistochrone for a system of forces for which* Q *is a free path—and the action and time in any arc of either, when it is described freely, are functions of the time and action respectively, in the same arc, when it is a brachistochrone.*

From this property Professor R. Townsend, *Quarterly Journal of Mathematics*, Vol. XIII., has shewn how to determine the intensity for parallel and concurrent forces for which given curves are brachistochrones.

For in the brachistochrone the velocity of description v for parallel forces must be proportional to the sine of the angle i between the directions of force and motion, and for concurrent forces must be proportional to the length of the perpendicular p from the centre of force in the direction of motion; provided that in addition the osculating plane at every point contains the direction of the force.

Hence

(a) For parallel forces, every curve (necessarily plane for brachistochronism in that case) for which $\sin^2 i = \phi(z)$,

where z is the ordinate in the direction of the force, is brachistochronous, under description with the velocity which would vanish with i, for the law of force $Z = \frac{1}{2} k^2 \phi'(z)$, k being any constant.

(*b*) For concurrent forces, every curve (necessarily plane for brachistochronism in that case also) for which $p^2 = \phi(r)$, where r is the radius vector from the centre of force, is brachistochronous under description with the velocity which would vanish with p, for the law of force $R = \frac{1}{2} k^2 \phi'(r)$, k being any constant.

In the following examples, given by Professor Townsend, the form of $\phi(z)$ or $\phi(r)$ being given, it is left as an exercise for the student to find the corresponding brachistochronous curve, the method of description, and the line of zero velocity;

$$(a) \qquad \phi(z) = \frac{z}{a},\ \frac{z^2}{a^2},\ \frac{a}{z},\ \frac{a^2}{z^2},\ 1 - \frac{z}{a},\ 1 - \frac{z^2}{a^2},$$

$$1 - \frac{a}{z},\ 1 - \frac{a^2}{z^2},\ 1 - \left(\frac{z}{a}\right)^{\frac{2}{3}},\ \left(\frac{z}{a}\right)^n,\ 1 - \left(\frac{z}{a}\right)^n,$$

$$\frac{z^2}{a^2 + z^2},\ \frac{a^4}{a^4 + z^4},\ \frac{a^2 z^2}{b^4 + (a^2 - b^2) z^2}.$$

$$(b) \qquad \phi(r) = ar,\ r^2 \sin^2 \alpha,\ \frac{r^3}{a},\ \frac{r^4}{a^2},\ \frac{r^6}{a^4},\ \frac{a^4}{r^2},\ \frac{r^n}{a^{n-2}},$$

$$r^2 - a^2,\ \pm m^2 (r^2 - a^2),\ \frac{a^2 r^2}{a^2 + r^2},\ \frac{(r^2 - b^2)^2}{a^2},\ \frac{a^2 b^2}{a^2 + b^2 - r^2},$$

$$\frac{b^2 r}{2a \pm r},\ \frac{r^2 (a^n - r^n)}{a^n}.$$

These examples will be found to contain most of the elementary brachistochrones that have been recognized, but given any curve the process is the same to determine the forces for which it is a brachistochrone for parallel or concurrent forces; $\phi(z)$ or $\phi(r)$ being determined from the property of the curve and $\phi'(z)$ or $\phi'(r)$ expressing the required law of intensity.

259. To solve the inverse problem, the determination of the brachistochrone from the law of force $\phi'(z)$ or $\phi'(r)$ supposed given, the differential equation between z and x or r and θ is immediately obtained from the general relation (a) or (b), but these differential equations can only be integrated in particular cases.

Thus if the force vary as the $(n-1)^{\text{th}}$ power of the distance, we have

$$a^n \sin^2 i = \pm (z^n - c^n),$$

or

$$a^{n-2} p^2 = \pm (r^n - c^n);$$

leading to the differential equations

$$\frac{dz}{dx} = \left(\frac{\pm a^n}{z^n - c^n} - 1\right)^{\frac{1}{2}},$$

or

$$\frac{dr}{rd\theta} = \left(\frac{\pm a^{n-2} r^2}{r^n - c^n} - 1\right)^{\frac{1}{2}},$$

which are not generally integrable in finite terms unless $c = 0$; the special case considered in examples 10, 11, and 21 given above.

260. Professor Townsend, *Quarterly Journal of Mathematics,* Vol. XIV., has also shewn how from the property (§ 185) that "if for the same velocity of description any curve, plane or twisted, be at once a free path for one system and a brachistochrone for another system of consecutive forces, the resultants of the two systems of forces must, at every point of the curve, be reflexions of each other, as regards both magnitude and direction, with respect to the current tangent at the point," cases of the free motion of a particle may be deduced from familiar cases of brachistochronous motion, and conversely.

Interesting applications are given of the principle to the comparison of the different methods of description in free and brachistochronous motion in well-known orbits, such as the parabola, the bifocal conics, the cycloid, catenary, &c.

Thus every bifocal conic being a free path for any combination of two forces emanating in similar or opposite

directions from the foci, and varying inversely as the square of the distance from its own focus, the velocity of description (real or imaginary) vanishing at each point (real or imaginary) of equal and opposite normal action of the forces; it follows that every bifocal conic, ellipse or hyperbola, is a brachistochronous path for any combination of forces emanating in similar or opposite directions from its two foci, and varying each inversely as the square of the distance from the other focus; the velocity of description (real or imaginary) vanishing at each point (real or imaginary) of equal and opposite normal action of the two forces.

261. *A particle moves in a plane, under an attraction directed to a point which moves in a given manner in the plane: to find the motion.*

Let x, y, ξ, η be the co-ordinates of the particle and point, at time t. ξ and η are given functions of t. Also let $P=f(r)$ be the acceleration due to the attraction at distance r. Then

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=-P\frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\\ \frac{d^2y}{dt^2}&=-P\frac{y-\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(1),$$

are the equations of motion.

The equations of *relative* motion are, of course,

$$\left.\begin{aligned}\frac{d^2(x-\xi)}{dt^2}&=-P\frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta)^2}}-\frac{d^2\xi}{dt^2}\\ \frac{d^2(y-\eta)}{dt^2}&=-P\frac{y-\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}-\frac{d^2\eta}{dt^2}\end{aligned}\right\}\ldots\ldots\ldots(2),$$

or, putting ξ_1, η_1, for the relative co-ordinates,

$$\left.\begin{aligned}\frac{d^2\xi_1}{dt^2}&=-P\frac{\xi_1}{\sqrt{\xi_1^2+\eta_1^2}}-\frac{d^2\xi}{dt^2}\\ \frac{d^2\eta_1}{dt^2}&=-P\frac{\eta_1}{\sqrt{\xi_1^2+\eta_1^2}}-\frac{d^2\eta}{dt^2}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(3).$$

These equations illustrate, in a particular case, the general theorem of § 26; as they contain, in addition to the terms due to the attraction of the fixed centre, the two known quantities $-\frac{d^2\xi}{dt^2}$ and $-\frac{d^2\eta}{dt^2}$, the components of acceleration of the centre *reversed*.

262. Ex. *Let the attraction vary directly as the distance.*

Here $P = \mu\sqrt{\xi_1^2 + \eta_1^2}$, and equations (3) of last section become

$$\left.\begin{aligned}\frac{d^2\xi_1}{dt^2} &= -\mu\xi_1 - \frac{d^2\xi}{dt^2}\\ \frac{d^2\eta_1}{dt^2} &= -\mu\eta_1 - \frac{d^2\eta}{dt^2}\end{aligned}\right\}\qquad\text{(4)},$$

which are easily integrated, in the form

$$\left.\begin{aligned}\xi_1 &= A\cos(\sqrt{\mu}t + B) - \frac{\left(\frac{d}{dt}\right)^2}{\left(\frac{d}{dt}\right)^2 + \mu}\xi\\ \eta_1 &= C\cos(\sqrt{\mu}t + D) - \frac{\left(\frac{d}{dt}\right)^2}{\left(\frac{d}{dt}\right)^2 + \mu}\eta\end{aligned}\right\}\qquad\text{(5)};$$

for particular values of ξ and η in terms of t.

Curiously enough, these equations shew that the form and position of the relative orbit are altered merely by shifting its centre, which is no longer at the centre of attraction.

As a particular case, suppose the centre of attraction to move with constant acceleration, α, parallel to a given direction, which may be taken as the axis of y. The centre of attraction will in general (Chap. IV.) describe a parabola, and the relative motion of the particle will be the same as in § 133, the centre of the ellipse or hyperbola being not at the

centre of attraction but at a distance $\frac{a}{\mu}$ from it in a line parallel to the axis of y.

Again, suppose the centre to move uniformly in a circle. We have

$$\xi = a\cos\omega t, \qquad \eta = a\sin\omega t,$$

and
$$\xi_1 = A\cos(\sqrt{\mu}t + B) - \frac{\omega^2 a}{\omega^2 - \mu}\cos\omega t,$$

$$\eta_1 = C\cos(\sqrt{\mu}t + D) - \frac{\omega^2 a}{\omega^2 - \mu}\sin\omega t,$$

$$\begin{aligned} x &= \xi + \xi_1 \\ &= A\cos(\sqrt{\mu}t + B) - \frac{\mu a}{\omega^2 - \mu}\cos\omega t, \end{aligned}$$

and
$$y = C\cos(\sqrt{\mu}t + D) - \frac{\mu a}{\omega^2 - \mu}\sin\omega t,$$

and the absolute path is therefore epitrochoidal.

263. *If the radius vector of a curve in space be at each instant parallel to the direction, and equal to the magnitude, of the velocity of a particle moving in any path; the curve is called the hodograph corresponding to the path* (§ 20).

The hodograph is evidently a plane curve if the path is so.

Let x, y, z be the co-ordinates of a point in the path, ξ, η, ζ those of the corresponding point of the hodograph; then evidently by the definition,

$$\left.\begin{aligned} \frac{dx}{dt} &= \xi \\ \frac{dy}{dt} &= \eta \\ \frac{dz}{dt} &= \zeta \end{aligned}\right\}.$$

Hence, if σ be the arc of the hodograph,

$$\frac{d\sigma}{dt} = \sqrt{\left\{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right\}}$$

$$= \sqrt{\left\{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2\right\}},$$

and the direction cosines of $\delta\sigma$ are proportional to

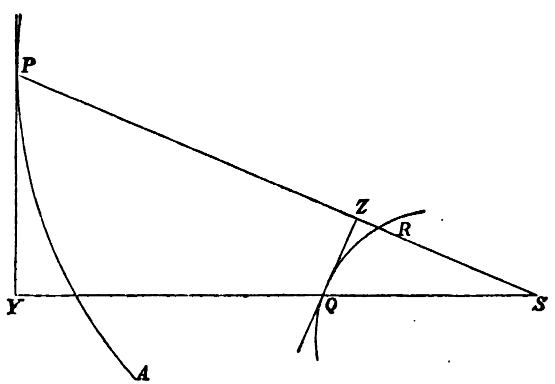
$$\frac{d^2x}{dt^2},\ \frac{d^2y}{dt^2},\ \frac{d^2z}{dt^2}.$$

Hence we see, as in § 20, that

The tangent to the hodograph at any instant is parallel to the resultant acceleration of the particle at the corresponding point of its path, and the velocity in it is equal to the acceleration of the particle.

264. The most important case of the hodograph being that corresponding to an orbit about a single centre of attraction we may deduce the above properties for that case in a somewhat different manner.

Let P be any point in PA, an arc of an orbit described about a centre of attraction S. Draw SY perpendicular to



the tangent at P, and take $SQ \,.\, SY = h$, then evidently SQ is equal to the velocity at P, and perpendicular to it in direction. Hence the locus of Q is the hodograph turned in its own plane through a right angle.

But we see that it is the polar reciprocal of PA with regard to a circle whose centre is S and radius $= \sqrt{h}$. Hence, by geometry, the tangent at Q is perpendicular to SP. This evidently corresponds to the first of the two general properties of the hodograph given in the last section.

Let $r, \theta, p, s, r', \theta', p', s'$ represent the usual quantities for corresponding points of the two curves; then if ρ' be the radius of curvature at Q, we have by the condition that QZ is perpendicular to SP,

$$\begin{aligned}\frac{ds'}{dt} &= \rho' \frac{d\theta}{dt} = r' \frac{dr'}{dp'} \frac{d\theta}{dt} \\ &= \frac{h}{p} \frac{d\frac{1}{p}}{d\frac{1}{r}} \frac{d\theta}{dt} = \frac{h}{p^3} \frac{dp}{dr} r^2 \frac{d\theta}{dt} \\ &= \frac{h^2}{p^3} \frac{dp}{dr} = P, \text{ (§ 139)},\end{aligned}$$

which proves the second property.

265. When the central attraction is inversely as the square of the distance, we have by § 264 for the arc of the hodograph,

$$\frac{ds'}{dt} = \frac{\mu}{r^2},$$

or $$\rho' = \frac{ds'}{d\theta} = \frac{ds'}{dt} \frac{dt}{d\theta} = \frac{\mu}{r^2} \frac{dt}{d\theta} = \frac{\mu}{h}.$$

Hence for all conic sections described about the focus the hodograph is a circle, as was first shewn by Hamilton.

This might have been shewn in another way, thus. In the fig. (§ 264) if PA be a portion of an ellipse or hyperbola of which S is the focus, the locus of Y is the auxiliary circle.

Hence evidently the locus of Q is a circle. If PA be a portion of a parabola of which S is the focus, the locus of Y is a straight line, and therefore that of Q is a circle passing through S.

Hence generally, the hodograph for any orbit about a centre of attraction inversely as the square of the distance, is a circle; about an internal point for an ellipse, an external point for a hyperbola, and about a point in the circumference for a parabola.

A purely analytical proof of the same theorem is easily given. If x, y be the co-ordinates of the planet, ξ, η those of a point in the hodograph, then

$$\xi = \frac{dx}{dt}, \quad \eta = \frac{dy}{dt}.$$

The equations of motion are

$$\frac{d^2x}{dt^2} = \frac{\mu x}{r^3} = \frac{\mu}{r^2}\cos\theta,$$

$$\frac{d^2y}{dt^2} = \frac{\mu y}{r^3} = \frac{\mu}{r^2}\sin\theta.$$

Hence, as usual,

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt} = h \quad\text{..............(1)},$$

and therefore

$$\frac{d^2x}{dt^2} = \frac{\mu}{h}\cos\theta\frac{d\theta}{dt} = \frac{\mu}{h}\frac{d}{dt}\left(\frac{y}{r}\right),$$

which gives, by integration,

$$\left.\begin{aligned} &\frac{dx}{dt} + A = \xi + A = \frac{\mu y}{hr}. \\ \text{Similarly}\quad &\frac{dy}{dt} + B = \eta + B = -\frac{\mu x}{hr}, \end{aligned}\right\} \quad\text{..............(2)},$$

and thence

$$(\xi + A)^2 + (\eta + B)^2 = \frac{\mu^2}{h^2},$$

proving that the hodograph is a circle.

Also, by eliminating $\frac{dx}{dt}$, $\frac{dy}{dt}$ among the three equations (1), (2), we get for the equation of the orbit

$$-h + Ay - Bx = \frac{\mu}{h} r,$$

which gives the focus and directrix property at once.

It is evident that that diameter of the circular hodograph which passes through the centre of force is divided by the centre of force in the same ratio as the axis major of the orbit is divided by the focus, and its length $= \frac{2\mu}{h}$.

266. The law of diffusion of heat and light from a calorific and luminous body is that of the inverse square of the distance. Hence an arc of the hodograph of a planet's orbit, which arc we have already seen to represent the *integral acceleration* due to the central attraction, represents also the entire amount of light or heat derived from the Sun during the passage through the corresponding arc of its orbit.

Ex. *Compare the amounts of light and heat received throughout their orbits by the Earth moving in a circle, and a comet moving in a parabola at the same perihelion distance.*

The hodographs are both circles, one about its centre, the other about a point in its circumference; but the diameter of the latter is $\sqrt{2}$ times the radius of the former (§ 149).

Hence their circumferences are $\sqrt{2} : 1$, or the Earth in its orbit receives in a revolution $\sqrt{2}$ times the amount of light and heat which the comet can receive in its whole path.

It is evident that the path apparently described by a fixed star, in consequence of the *Aberration* of light, is the Hodograph of the Earth's orbit, and is therefore a circle in a plane parallel to the ecliptic, and of the same dimensions for all stars.

267. Sir W. R. Hamilton enunciates (*Lectures on Quaternions*, p. 614) the following proposition:

If two circular hodographs, having a common chord, which passes through, or tends to, a common centre of force, be both cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.

It is evident from (§ 265), that the two orbits are conic sections of the same species, and with equal major axes.

Also, every circle which cuts both hodographs perpendicularly must have its centre on the common chord. Let the

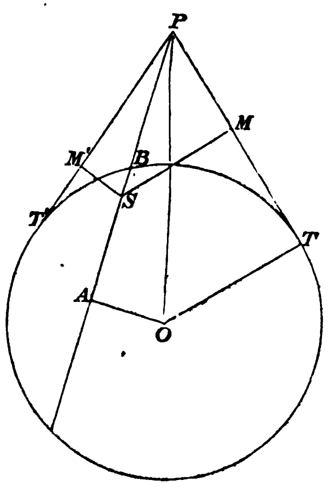


figure represent one of the hodographs, S being the centre of force, and ABP the common chord. Take any point P and draw the tangents PT, PT'. We proceed to investigate the difference of the times of hodographically describing TT' and the corresponding arc for a position of P slightly shifted along AP.

Draw OA perpendicular to AP. Let $OT=a$, $AB=b$, $OA=c$, $SP=r$, $SM=\varpi$, $SM'=\varpi'$, $PO=q$, $PA=r'$, and $PT=PT'=\tau$. If P be moved through a space δr, the increase of the angle PSM which is the angle vector in the orbit, is $\dfrac{\varpi\delta r}{r\tau}$ nearly. But the corresponding radius vector in the orbit is $\dfrac{h}{\varpi}$ (§ 264) and therefore the time of hodographically describing the small arc at T is

$$\delta t=\frac{1}{h}\frac{h^2}{\varpi^2}\frac{\varpi\delta r}{r\tau}=\frac{\mu\delta r}{r\tau}\frac{1}{\varpi a}.\quad (\S\ 265.)$$

Hence the whole change produced in the time of hodographically describing the arc TT' by shifting P is

$$\frac{\mu\delta r}{r\tau}\left(\frac{1}{a\varpi}+\frac{1}{a\varpi'}\right)=\frac{2\mu r'\delta r}{b^2r^2\tau}.$$

[This is easily seen, if we notice that by the figure

$$\left.\begin{matrix}\varpi\\ \varpi'\end{matrix}\right\}=r\sin\left\{\sin^{-1}\frac{a}{q}\pm\sin^{-1}\frac{c}{q}\right\}.]$$

Now this is the same for both hodographs, and, as the arc TT' vanishes for each when P is at B, we have the proposition.

It will readily be seen that this is in substance the same as Lambert's Theorem (§ 168).

268. We now take an instance of the determination, from the hodograph and the law of its description, of the curve described and the forces acting.

The hodograph is a circle described with constant angular velocity about a point in its circumference, find the original path and the circumstances of its description.

Here we have in the hodograph,

$$\rho=a\cos\theta,$$
$$\theta=\omega t;$$

therefore in the path

$$\frac{dx}{dt} = \rho \cos\theta = a \cos^2 \omega t,$$

$$\frac{dy}{dt} = \rho \sin\theta = a \cos\omega t \sin\omega t.$$

Integrating and properly adapting the constants, as they affect only the position of the origin,

$$x = \frac{a}{4\omega}(2\omega t + \sin 2\omega t),$$

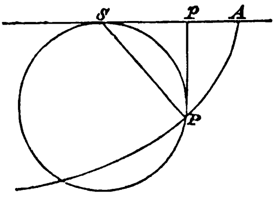
$$y = \frac{a}{4\omega}(1 - \cos 2\omega t).$$

Now the equations of a cycloid are

$$x = A\,(\phi + \sin\phi),$$
$$y = A\,(1 - \cos\phi);$$

hence the path is a cycloid; and, since $2\omega t = \phi$, the direction of motion revolves uniformly. The particle moves under a constant force perpendicular to the base of the cycloidal constraining curve, and the velocity at any point is that due to the distance from the base, which is the brachistochrone of § 180. The converse is easily proved.

Geometrically thus, if AP be the cycloid described by the point P of the circle SP rolling uniformly on the line AS, the velocity at P is proportional to SP, and the direction of motion is perpendicular to SP. Hence the hodograph (turned

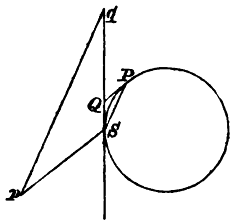


through a right angle in its own plane) may be represented by the circle SP, described with uniform angular velocity

about the point S. That the motion is due to constant acceleration perpendicular to AS is obvious from the fact that, if Pp be drawn perpendicular to AS, $SP^2 \propto Pp$.

269. *If the orbit be central, and be a circle described about a point in its circumference, the hodograph is a parabola described about the focus with angular velocity proportional to the radius vector.*

For, if S be the centre of attraction, P the particle in its circular orbit, p the corresponding point of the hodograph: qp, the tangent to the hodograph at p, must be parallel to SP; and, therefore, if SQq be the tangent at S, the triangle pSq (being similar to PSQ) is isosceles. Thus the locus of p is a parabola, for its tangent, pq, is equally inclined to the radius-vector Sp, and to the fixed line Sq. Also the angular velocity of Sp, being the same as that of PQ, is double that



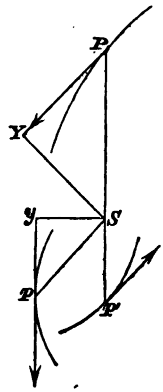
of SP, and is, therefore, inversely as SP^2. But the length of Sp is inversely as the perpendicular from S upon PQ, i.e. inversely as SP^2.

Or immediately, the pedal of a circle with respect to a point on the circumference is a cardioid, and the hodograph, which is the inverse of the pedal, is therefore a parabola.

270. *The only central orbits whose hodographs also are described as central orbits, are those in which the acceleration varies directly as the distance from the centre.*

Let S be the centre, P any point in the path, p the corresponding point in the hodograph, p' that in the hodograph

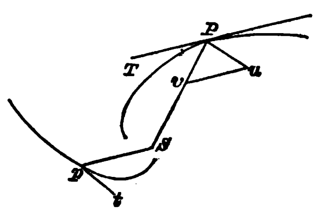
of the hodograph. Then Sp' is parallel to the tangent at p, which again is parallel to SP. Hence PSp' is a straight line.



Also, since p belongs (by hypothesis) to a central orbit, the tangent at p' is parallel to Sp, i.e. to the tangent at P. Hence the locus of p' is similar to that of P, and therefore Sp' is *proportional to* Sp. But Sp' represents the acceleration at P. Hence the proposition.

271. *A point describes a logarithmic spiral with constant angular velocity about the pole; find the acceleration.*

Since the angular velocity of SP and the inclination of this line to the tangent are each constant, the linear velocity



of P is as SP. Take a length PT, equal to $e.SP$, to represent it. Then the hodograph, the locus of p, where Sp is parallel,

and equal, to PT, is evidently another logarithmic spiral similar to the former, and described with the same constant angular velocity. Hence pt, the acceleration required, is equal to $e.Sp$, and makes with Sp an angle equal to SPT. Hençe, if Pu be drawn parallel and equal to pt, and uv parallel to PT, the whole acceleration Pu may be resolved into Pv and vu; and Pvu is an isosceles triangle, whose base angles are each equal to the angle of the spiral. Hence Pv and vu bear constant ratios to Pu, and therefore also to SP or PT.

The acceleration, therefore, is composed of a centripetal acceleration proportional to the distance, and a tangential retardation proportional to the velocity.

And, if the resolved part of P's motion parallel to any line in the plane of the spiral be considered, it is obvious that in it also the acceleration will consist of two parts—one directed towards a point in the line (the projection of the pole of the spiral), and proportional to the distance from it, the other proportional to the velocity, but retarding the motion.

Hence a particle which, unresisted, would have a simple harmonic motion, has, when subject to resistance proportional to its velocity, a motion represented by the resolved part of the spiral motion just described.

If α be the angle of the spiral, ω the angular velocity of SP, we have evidently $PT.\sin\alpha = SP.\omega$.

Hence

$$Pv = Pu = pt = \frac{PT^2}{SP} = \frac{\omega}{\sin\alpha} PT = \frac{\omega^2}{\sin^2\alpha} SP = n^2 . SP \text{ (suppose)}$$

and $$vu = 2Pv . \cos\alpha = \frac{2\omega\cos\alpha}{\sin\alpha} PT = 2k . PT \text{ (suppose)}.$$

Thus the central acceleration at unit distance is $n^2 = \dfrac{\omega^2}{\sin^2\alpha}$,

and the coefficient of resistance is $2k = \dfrac{2\omega\cos\alpha}{\sin\alpha}$.

The time of oscillation is evidently $\frac{2\pi}{\omega}$; but, if there had been no resistance, the properties of simple harmonic motion shew that it would have been $\frac{2\pi}{n}$; so that it is increased by the resistance in the ratio $\operatorname{cosec} \alpha : 1$, or $n : \sqrt{n^2 - k^2}$.

The rate of diminution of SP is evidently

$$PT \,.\, \cos \alpha = \frac{\omega \cos \alpha}{\sin \alpha} SP = kSP;$$

that is, SP diminishes in geometrical progression as time increases, the rate being k per unit of time per unit of length. By an ordinary result of arithmetic (compound interest payable every instant) the diminution of $\log . SP$ in unit of time is k.

This process of solution is only applicable to resisted harmonic vibrations when n is greater than k. When n is not greater than k the auxiliary curve can no longer be a logarithmic spiral, for the moving particle never describes more than a finite angle about the pole. A curve, derived from an equilateral hyperbola, by a process somewhat resembling that by which the logarithmic spiral is deduced from a circle, must be introduced; and then the geometrical method ceases to be simpler than the analytical one, so that it is useless to pursue the investigation farther, at least from this point of view.

These geometrical results may easily be deduced by the principles of the preceding chapter, which give at once for the rectilinear motion the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + n^2x = 0.$$

See *Proc. R. S. E.* for farther illustrations.

EXAMPLES.

(1) Investigate the differential equation of the path of a particle in a plane

$$2X = \frac{d}{dx}\left(\frac{Y - X\dfrac{dy}{dx}}{\dfrac{d^2y}{dx^2}}\right).$$

(2) A particle slides down an inverted cycloid from rest at the cusp; shew that the whole acceleration at any instant is g, and that its direction is towards the centre of the generating circle. Prove also that the motion of the particle will be produced by rolling the generating circle on the under side of a horizontal straight line with velocity $\sqrt{ga}$, where a is the radius of the generating circle.

(3) If a curve whose equation is $y = f(x)$ is described freely by a particle under potential V, and if the same curve can be described freely under potential

$$V + \phi\{y - f(x)\},$$

prove that the curve must be a cycloid.

(4) If a particle move on a rough inclined plane, prove that

$$\sqrt{\rho\rho'}\cos^3\theta = r,$$

where ρ, ρ' are the radii of curvature of the path at the two points where the tangents are inclined at an angle θ to the horizon, and r is the radius of curvature at the highest point.

(5) A particle is projected up a rough inclined plane. Shew that the intrinsic equation to the curve described is

$$s\sin\alpha = \frac{v^2}{g}\left(\tan\frac{\beta}{2}\right)^{2\mu\cot\alpha}\sin^2\beta\int_\beta^\phi\left(\cot\frac{\phi}{2}\right)^{2\mu\cot\alpha}\operatorname{cosec}^3\phi\, d\phi,$$

where v = velocity of projection and β = angle between direction of projection and the line of greatest slope.

(6) A particle moves under two constant forces in the ratio of 9 to 1 whose directions rotate in opposite directions with constant angular velocities in the ratio of 3 to 1; prove that under certain initial conditions the path of the particle will be a closed curve of the form represented by the equation $r = a \cos 2\theta$.

(7) A particle is attracted by an infinite straight line AB with intensity which is inversely proportional to the cube of the distance of the particle from the line. The particle is projected with the velocity from infinity from a point P at a distance a from the nearest point O of the line in a direction perpendicular to OP, and inclined at the angle α to the plane AOP. Prove that the particle is always on the sphere of which O is the centre; that it meets every meridian line through AB at the angle α; and that it reaches the line AB in the time $\dfrac{a^2}{\sqrt{\mu}\cos\alpha}$, μ being the strength of the attraction.

(8) Shew that if a material particle move under any conservative system of forces, the projection of the principal radius of curvature of its path at any point on the direction of the resultant force at that point is

$$\frac{v}{\left\{\left(\dfrac{dv}{dx}\right)^2 + \left(\dfrac{dv}{dy}\right)^2 + \left(\dfrac{dv}{dz}\right)^2\right\}^{\frac{1}{2}}},$$

v denoting the velocity of the particle.

(9) If r be the radius vector of any point on a curve, p the perpendicular from the origin on the tangent at that point, s the length of the arc, and $\phi(r)$ any function of r, prove that, if $\int \phi(r)\,ds$ (the integral being taken between finite limits) be a maximum or minimum, then $\phi(r) \propto \dfrac{1}{p}$.

(10) Jets of water escape horizontally from orifices along a generating line of a vertical cylinder kept always full. Shew

that (to axes inclined 45° to the vertical) the equation of the lines of equal Action for unit mass of water is of the form

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$$

Shew also that the line of equal time for particles of water issuing simultaneously from the orifices is the free path of the water which leaves the vessel by an orifice at a depth below the surface due to that time.

(11) A number of particles fall down the arcs of vertical circles which have their highest points and the tangents at them in common, from rest at the highest point. Prove that the equation of the line of equal Action is

$$r^3 = \frac{k^2 \sin^3\theta}{(1 - \cos\theta)^2},$$

r being measured from the highest point of the circles.

(12) Of all the different sets of paths along which a conservative system may be guided to move from one configuration to another, with the sum of its potential and kinetic energies equal to a given constant, that one for which the Action is a minimum is such that the system will require only to be started with the proper velocities to move along it unguided.

Shew that, if APB be a projectile's path, AB the latus rectum, AT, TB tangents at A and B, the Action will be the same for the free path APB as for the constrained path ATB.

(13) A particle attracted towards a fixed centre, with intensity varying as the distance from that centre, is projected with a given velocity at right angles to the line joining the point of projection with the centre so as to describe an ellipse. Prove that its Action in one revolution will be greater than it would have been if it had been constrained to describe the circle round the same centre of attraction having for radius the distance of projection, the velocity of projection being the same as before.

Is this result inconsistent with the principle of "Least Action"?

(14) If a particle move in the brachistochrone between two given points under gravity on a smooth surface of revolution of which the axis is vertical, prove that the area swept out by the projection of the ordinate on a horizontal plane is proportional to the Action.

(15) The velocity of a particle in a central orbit varies as $\frac{1}{r^2}$. Apply the principle of Least Action to find the orbit, and thence the law of attraction. Deduce the same results from the Conservation of Energy.

(16) If $u = F(x, y, z, a, b, k) + c$ is a complete solution of the equation

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 = 2(U+k),$$

where U is a given function of x, y, z, and k is a constant; prove that

$$\left(\frac{du}{da}\right) = \alpha, \quad \left(\frac{du}{db}\right) = \beta,$$

are the equations to any orthogonal trajectory of the system of surfaces, for points on each of which u has a constant value, and that, if points move along these trajectories with velocities, which in any position are equal to the value of $\sqrt{2(U+k)}$ at that point, their position at any time is determined by the equation

$$\left(\frac{du}{dk}\right) = t + \tau,$$

where τ is an arbitrary constant.

(17) Prove that every curve, plane or twisted, for which $s^2 = \phi(x, y, z)$, where s is the length of any arc of it AP measured from a fixed point A, and x, y, z the rectangular co-ordinates of its variable extremity P, is tautochronous with

respect to the fixed point for the force, or system of forces, whose components parallel to the co-ordinate axes are

$$X = -\frac{1}{2}k^2\frac{d\phi}{dx}, \qquad Y = -\frac{1}{2}k^2\frac{d\phi}{dy}, \qquad Z = -\frac{1}{2}k^2\frac{d\phi}{dz},$$

k being any constant.

(18) Prove that a rhumb line on the surface of a sphere is tautochronous with respect to either pole, for a force acting radially from, or perpendicularly from the tangent plane at, the opposite pole, and varying in either case directly as the length and inversely as the sine of the spherical distance from the original pole. (*Prof. Townsend.*)

(19) Prove that for parallel forces, every curve, plane or twisted, for which $s^2 = \phi(z)$, where z is the ordinate in the direction of force, is tautochronous with respect to the origin of s, for the law of force $Z = -\frac{1}{2}k^2\phi'(z)$, k being any constant.

Prove that for concurrent forces every curve, plane or twisted, for which $s^2 = \phi(r)$, where r is the vector from the centre of force, is tautochronous with respect to the origin of s, for the law of force $R = -\frac{1}{2}k^2\phi'(r)$, k being any constant.

Interpret the curves

$$s^2 = z^2\sec^2\alpha,\ s^2 = r^2\sec^2\alpha,\ s^2 = 4a(a-z),\ s^2 = 4a(a-r),$$

$$s^2 = z^2 - a^2,\ s^2 = r^2 - a^2,\ s^2 = \pm m^2(r^2 - a^2),$$

$$2as = z^2 - a^2,\ 2as = r^2 - a^2,\ \frac{s}{a} = \cos^{-1}\frac{z}{a},$$

$$s = \cos^{-1}\frac{r}{a}.$$ (*Prof. Townsend.*)

(20) A particle under a system of forces describes their

tautochrone in a time T. Shew that the action in a complete oscillation is

$$\frac{2\pi^2c^2}{T},$$

where $2c$ is the length of the arc described.

(21) Shew that the pressure of a particle of mass m on a tautochrone under any conservative system of forces is

$$mF\left\{\sin\phi + \frac{s^2 - s_0^2}{\rho s}\cos\phi\right\},$$

where ρ is the radius of curvature at the point, ϕ the inclination of the resultant force mF to the tangent, and s, s_0 the distances measured along the curve of the point and starting-point from the point where the times of fall are equal.

(22) A particle, under a central attraction, the acceleration due to which at a distance r is $\frac{\mu r}{(a^2+r^2)^2}$, a being a constant, is projected from a given point with the velocity from infinity; prove that the form of the groove, in which it must move in order to arrive at another given point in the shortest possible time, is an hyperbola whose centre coincides with the centre of attraction.

(23) A body is such that it is its own level surface. Shew that the brachistochrone from any point to the body is the line of force passing through the point.

(24) If $\theta, \phi, \psi\dots$ be the generalized co-ordinates of a conservative system, T its kinetic energy, and if $\theta, \phi, \psi\dots$ be supposed to be expressed explicitly in terms of t and arbitrary constants, and if Δ, δ be the symbols of two independent variations of the arbitrary constants, prove that

$$\Delta\theta\,.\,\delta\frac{dT}{d\dot\theta} + \Delta\phi\,.\,\delta\frac{dT}{d\dot\phi} + \Delta\psi\,.\,\delta\frac{dT}{d\dot\psi} + \dots$$

$$-\,\delta\theta\,.\,\Delta\frac{dT}{d\dot\theta} - \delta\phi\,.\,\Delta\frac{dT}{d\dot\phi} - d\psi\,.\,\Delta\frac{dT}{d\dot\psi} - \dots$$

is independent of t; $\dot\theta$, $\dot\phi$, $\dot\psi$ ... denoting $\frac{d\theta}{dt}$, $\frac{d\phi}{dt}$, $\frac{d\psi}{dt}$ ... respectively. Illustrate this by reference to the motion (1) of a projectile, (2) of a system of particles attracting each other with intensities varying as the distance.

(25) Shew that the amounts of heat and light received by a planet in one revolution are each inversely as the square root of the latus rectum of its orbit.

(26) If P and Q be the accelerations along the tangent and normal to the path of a particle, and ψ the angle the tangent makes with a fixed line, the equation of the hodograph will be

$$r = a\epsilon^{\int \frac{P}{Q} d\psi},$$

where a is a constant.

(27) Find analytically a central orbit whose form and mode of description correspond with those of the hodograph of another central orbit.

Shew that there is but one law of central attraction for which this is possible except, of course, in the case of the original orbit being a circle about its centre, when *any* law may obtain. § 270.

(28) If P, P' be the central accelerations for an orbit and its hodograph, prove that

$$PP' = \frac{h'^2}{h^2} rr'.$$

(29) Shew that the central acceleration necessary to make a particle describe the hodograph of a central orbit is inversely proportional to the normal acceleration at the corresponding point of the orbit.

(30) Shew that in the hodograph of a central orbit whose acceleration is $f(r)$, the curvature varies inversely as $r^2 f(r)$.

(31) When the hodograph is a straight line described with constant velocity, the path is the trajectory of an unresisted projectile.

(32) When it is a straight line described with constant angular velocity about a point, the path is the catenary of uniform strength

$$\epsilon^{\frac{y}{k}} = \sec\frac{x}{k},$$

and the acceleration is parallel to y and varies as the square of either of these equal quantities.

(33) Prove that the area swept out by the radius vector of a projectile, drawn from its point of projection, varies as the cube of the time of describing it.

(34) If the hodograph be a circle about a point in its circumference, and if θ being the angle which the radius vector makes with the diameter, the angular velocity be given by

$$\frac{d\theta}{dt} = \frac{k}{\sqrt{(\epsilon^{2kt}-1)}};$$

shew that the path is a cycloid with its vertex upwards, and that the velocity at any point is that due to a fall from the tangent at the vertex.

(35) If a circle be described under a constant acceleration not tending to the centre, the hodograph is a lemniscate.

(36) A particle is moving in a parabolic orbit so that the velocity of its recession from the focus is constant; ascertain the form of the hodograph of the particle.

(37) The hodograph of an orbit is a parabola whose ordinate increases with constant velocity. Prove that the orbit is a semi-cubical parabola.

(38) A straight rod, the ends of which are moveable along two perpendicular straight lines in one plane, revolves with a constant angular velocity. Prove that the hodographs of the paths of its points are ellipses enveloped by a hypocycloid.

(39) Define the hodograph of a point moving in any manner; and find its equation, for a point on the circumference of a wheel, which rolls uniformly within the circumference of a fixed wheel of four times its radius.

(40) A smooth elliptic tube is placed with its major axis vertical and a particle allowed to slide down it, starting from rest at the highest point; shew that the hodograph is given by the equation

$$r = 2\sqrt{ga}\sin\frac{1}{2}\left\{\cot^{-1}\left(\frac{a}{b}\cot\theta\right)\right\}..$$

(41) Prove that the hodograph of a catenary, described freely under an acceleration parallel to the axis, is a straight line described with velocity proportional to that in the catenary.

(42) Prove that the hodograph of a central orbit is its reciprocal polar with respect to the centre of attraction.

Prove that the equation of the hodograph of a cardioid described under an attraction to the cusp may be put in the form

$$r\sin^3\frac{\theta}{3} = a.$$

(43) A lemniscate whose equation is $r^2 = a^2\cos 2\theta$ is placed with the initial line vertical, and a particle is constrained to move on it, moving from rest at the pole; prove that the hodograph is defined by the equation

$$r^4 = c^4\cos\frac{\pi+2\phi}{3}\cos^2\frac{\pi+2\phi}{6},$$

where c is a constant.

(44) If a particle move under a constant acceleration which is initially normal, and which, when the direction of motion of the particle has turned through an angle ϕ, has turned through an angle 2ϕ in the opposite direction; prove that the equation of the hodograph is

$$r^3\cos 3\theta = c^3,$$

and the equation of the orbit is

$$r^{\frac{3}{2}} \cos \frac{3}{2}\theta = a^{\frac{3}{2}}.$$

(45) Two particles are describing free paths in one plane which are hodographs to one another; if the particles be always at corresponding points, prove that the paths must be conic sections, and find the nature of the forces acting on the particles.

(46) The resistance of the air being supposed to vary as the cube of the velocity, shew that the hodograph of a projectile is

$$x^3 + 3xy^2 = ay^3 + b,$$

the axis of x being vertical.

(47) A particle moves freely under a force whose direction is always parallel to a fixed plane, and describes a curve which lies on a right circular cone, and crosses the generating lines at a constant angle; prove that its hodograph is a conic section.

CHAPTER IX.

IMPACT.

272. WE come next to the consideration of the effects of a class of actions which cannot be treated by the methods employed in the preceding chapters. These are called *Impulsive* actions, and are such as arise in cases of collision; lasting (in the case of bodies of moderate dimensions) for an exceedingly short time only, and yet producing finite changes of momentum. Hence, in dealing with the immediate effects of such impulses, *finite* forces acting along with them need not be considered.

When two balls of glass or ivory impinge on one another, no doubt there goes on a very complicated operation during the brief interval of contact. First, the portions of the surfaces immediately in contact are disfigured and compressed until the molecular reactions thus called into action are sufficient to resist farther distortion and compression. At this instant it is evident that the points in contact are moving with the same velocity. But, most solids being endowed with a certain degree of elasticity of form, the balls tend to recover their spherical form, and an additional pressure is generated; proportional, as Newton found by experiment, to that exerted during the compression. The coefficient of proportionality is a quantity determinable by experiment, and may be conveniently termed the *Coefficient of Restitution.* It is always less than unity.

The method of treating questions involving actions of this nature will be best explained by taking as an example the case of *direct impact of one spherical ball on another;* first, when the balls are inelastic. Again, when their coefficient of restitution is given.

And it is evident that in the case of direct impact of smooth or non-rotating spheres we may consider them as mere particles, since everything is symmetrical about the line joining their centres.

273. Suppose that a sphere of mass M, moving with a velocity v, overtakes and impinges on another of mass M', moving in the same direction with velocity v'; and that at the instant when the mutual compression is completed, the spheres are moving with a common velocity V. Let P be the pressure between them at any time t during the compression, and τ the time during which compression takes place, then we have

$$M(v - V) = \int_0^\tau P dt = R, \text{ suppose,}$$

$$M'(V - v') = \int_0^\tau P dt = R;$$

whence $V = \dfrac{Mv + M'v'}{M + M'}$, and $R = \dfrac{MM'}{M + M'}(v - v')$.

From these results we see that the whole momentum after impact is the same as before, and that the common velocity is that of the centre of inertia before impact. Had the balls been moving in opposite directions, v' would have been negative, and in that case we should have

$$V = \frac{Mv - M'v'}{M + M'}, \text{ and } R = \frac{MM'}{M + M'}(v + v').$$

From the first of these results it appears that both balls will be reduced to rest if

$$Mv = M'v';$$

that is, if their momenta were originally equal and opposite.

This is the complete solution of the problem if the balls be inelastic, or have no tendency to recover their original form after compression.

274. If the balls be elastic, there will be generated, by their tendency to recover their original forms, an additional pressure proportional to R.

Let e be the coefficient of restitution, v_1, v_1', the velocities of the balls when finally separated. Then, as before,

$$M(V - v_1) = eR,$$
$$M'(v_1' - V) = eR;$$

whence

$$Mv_1 = M\frac{Mv + M'v'}{M+M'} - e\frac{MM'}{M+M'}(v - v'),$$

and

$$v_1 = \frac{(M - eM')\,v + M'(1+e)\,v'}{M+M'} = v - \frac{M'}{M+M'}(1+e)(v - v'),$$

with a similar expression for v_1'.

A rather singular result may easily be deduced from the last formula. Suppose $M = M'$, $e = 1$, that is, let the balls be of equal mass, and their coefficient of restitution unity (or, in the usual, but most misleading phraseology, "Suppose the balls to be *perfectly elastic*"); then in this case

$$v_1 = v', \text{ and similarly } v_1' = v,$$

or the balls, whatever be their velocities, interchange them, and the motion is the same as if they had passed through one another without exerting any mutual action whatever.

275. The only other case which we can treat in the present work is that of oblique impact when the balls are spherical and perfectly smooth, for in rough and non-spherical balls rotations are generated and the motion of each ball requires to be treated as that of a rigid body.

The simplest case is that of *a particle impinging with given velocity, and in a given direction, on a smooth fixed plane.*

Suppose the plane of the particle's motion to be taken as that of reference; its trace on the given plane as the axis of x, and the point at which the impact takes place, as origin.

The impulsive reaction of the plane will be perpendicular to it, since it is smooth. Let this be called R; and let the velocity of the particle be resolved into two, v_x, v_y, respectively parallel to the axes. For the first part of the impact

$$M(v_x - v_x') = 0,$$

$$M(v_y - v_y') = R.$$

But v_y', being the common velocity of the plane and ball, is evidently zero; hence

$$v_x' = v_x, \quad v_y' = 0,$$

or, the velocity parallel to the plane is unchanged, while that perpendicular to it is destroyed. So far for an inelastic ball. If the ball be elastic, let v_x'', v_y'' be the final velocities, then

$$M(v_x' - v_x'') = 0,$$

$$M(v_y' - v_y'') = eR.$$

These equations give

$$v_x'' = v_x' = v_x,$$

shewing that the velocity parallel to the plane is unaffected; and

$$Mv_y'' = -eR = -eMv_y,$$

or,

$$v_y'' = -ev_y,$$

that is, the velocity perpendicular to the plane is reversed in direction, and diminished in the ratio $e : 1$.

If we designate by the name of angle of incidence the inclination of the original direction of the ball's motion to the normal to the plane, and by that of angle of reflexion the angle made with the same line by the path after impact; then denoting the total velocities before and after impact by V and V'', and these angles by θ, ϕ respectively, we have

$$V \sin\theta = v_x, \quad V'' \sin\phi = v_x'',$$

$$V \cos\theta = v_y, \quad V'' \cos\phi = v_y'';$$

and the previous results give at once

$$\left.\begin{aligned} e\cot\theta &= \cot\phi \\ V'' &= V\frac{\sin\theta}{\sin\phi} \end{aligned}\right\}.$$

Of course these results are applicable to cases of impact on any smooth surface; by making the legitimate assumption

that the impact, and its consequences as regards the motion of the ball, would be the same if for the surface its tangent plane at the point of contact were substituted.

276. *Two smooth spheres, moving in given directions and with given velocities, impinge; to determine the impulse and the subsequent motion.*

Let the masses of the spheres be M, M'; their velocities before impact v and v', and let the original directions of motion make with the line which joins the centres at the instant of impact, angles α, α'. These angles may easily be calculated from the data, if the radii of the spheres be given.

It is evident that, since the spheres are smooth, the entire impulse takes place in the line joining the centres at the instant of impact, and that therefore the future motion of each sphere will be in the plane passing through this line and its original direction of motion.

Let R be the impulse, e the coefficient of restitution; then since the velocities in the line of impact are $v \cos \alpha$ and $v' \cos \alpha'$, we have for their final values v_1, v_1', after restitution, by § 274, the expressions

$$v_1 = v \cos \alpha - \frac{M'}{M+M'}(1+e)(v \cos \alpha - v' \cos \alpha'),$$

$$v_1' = v' \cos \alpha' + \frac{M}{M+M'}(1+e)(v \cos \alpha - v' \cos \alpha'),$$

and the value of R is

$$\frac{MM'}{M+M'}(1+e)(v \cos \alpha - v' \cos \alpha').$$

Hence, the sphere M has finally a velocity v_1 in the line joining the centres, and a velocity $v \sin \alpha$ in a known direction perpendicular to this, namely in the plane through this and its original direction of motion. And similarly for the sphere M'. Thus the impact is completely determined.

277. Recurring to the equations in § 273, we have

$$M(v-V)=R,$$
$$M'(V-v')=R,$$

and, eliminating V,

$$R=\frac{MM'}{M+M'}(v-v') \dots\dots\dots\dots\dots\dots(1).$$

Hence, if e be the coefficient of restitution, v_1, v_1' the final velocities,

$$\left.\begin{aligned} v_1 &= v-\frac{R(1+e)}{M} \\ v_1' &= v'+\frac{R(1+e)}{M'} \end{aligned}\right\} \dots\dots\dots\dots\dots\dots(2).$$

Hence, $Mv_1+M'v_1'=Mv+M'v'$, whatever e be, or there is no momentum lost. This is, of course, a direct consequence of the Third Law of Motion.

Again, $\frac{1}{2}Mv_1^2+\frac{1}{2}M'v_1'^2=\frac{1}{2}Mv^2+\frac{1}{2}M'v'^2$

$$-R(1+e)(v-v')+\tfrac{1}{2}R^2(1+e)^2\frac{M+M'}{MM'}$$

$$=\tfrac{1}{2}Mv^2+\tfrac{1}{2}M'v'^2-\tfrac{1}{2}R^2(1-e^2)\frac{M+M'}{MM'}$$

$$=\tfrac{1}{2}Mv^2+\tfrac{1}{2}M'v'^2-\tfrac{1}{2}(1-e^2)\frac{MM'}{M+M'}(v-v')^2.$$

The last term of the right-hand side is therefore the kinetic energy apparently destroyed by the impact. When $e=0$, its magnitude is greatest and equal to $\frac{MM'}{M+M'}\frac{1}{2}(v-v')^2$. When $e=1$ its magnitude is zero, that is, when the coefficient of restitution is unity no kinetic energy is lost.

The kinetic energy which appears to be destroyed in any of these cases is, as we see from § 78*, only transformed—partly it may be into heat, partly into sonorous vibrations, as in the impact of a hammer on a bell. But, in spite of this, the elasticity may be *perfect.* Hence the absurdity of the common designation alluded to in § 274.

Also by (2),

$$v_1' - v_1 = v' - v + R(1+e)\frac{M+M'}{MM'}$$

$$= e(v - v'), \text{ by (1).}$$

Hence the velocity of separation is e times that of approach. These results may easily be extended to the more general case of § 276.

The case of a *rough* sphere cannot be treated here, inasmuch as it involves the Dynamics of a Rigid Body, and this is beyond our professed limits.

278. We proceed to some special problems illustrating the subject of impact.

To one end of a chain, lying in a given curve on a smooth horizontal plane, a given impulsive tension is applied in the direction of the tangent at that end; it is required to find the impulsive tension at any other point of the chain.

Let this be T at a point of the chain whose co-ordinates are x, y; and let the initial velocities of that point, parallel to the axes, be v_x, v_y; then, μ being the mass per unit of length of the chain, we have the following equations:

$$\left.\begin{aligned}\frac{d}{ds}\left(T\frac{dx}{ds}\right) &= \mu v_x\\ \frac{d}{ds}\left(T\frac{dy}{ds}\right) &= \mu v_y\end{aligned}\right\}\text{......................(1).}$$

The geometrical condition is to be determined as follows. The chain being inextensible, the length of an element δs is invariable, therefore the velocities of its two extremities resolved along the element must be the same. This gives evidently

$$\frac{dv_x}{ds}\frac{dx}{ds} + \frac{dv_y}{ds}\frac{dy}{ds} = 0 \text{......................(2).}$$

Or, if v_s, v_ρ be the velocities generated at any point, in the direction of the tangent and normal, we have at once

$$\left.\begin{array}{l} \dfrac{dT}{ds} = \mu v_s \\ \dfrac{T}{\rho} = \mu v_\rho \end{array}\right\} \text{.........................(3)};$$

and the kinematical condition furnished by the inextensibility of the chain

$$\frac{dv_s}{ds} = \frac{v_\rho}{\rho} \text{..............................(4)}.$$

If ϕ be the angle the instantaneous direction of motion at any point makes with the tangent,

$$\tan\phi = \frac{v_\rho}{v_s} = \frac{T}{\rho\dfrac{dT}{ds}} \text{..............................(5)}.$$

By the elimination of v_s and v_ρ we obtain

$$\frac{d}{ds}\left(\frac{1}{\mu}\frac{dT}{ds}\right) - \frac{T}{\mu\rho^2} = 0 \text{ ....(6)},$$

the general differential equation of the impulsive tension at any point.

This of course cannot be integrated unless the initial form of the chain is known, i.e. unless μ and ρ are given in terms of s.

Another method of solution is given in Thomson and Tait's "Natural Philosophy," §§ 310, 311, where it is shewn that in such a case the chain takes the least possible kinetic energy; this gives, by equations (3),

$$\delta\int\left\{\left(\frac{1}{\mu}\frac{dT}{ds}\right)^2 + \left(\frac{T}{\mu\rho}\right)^2\right\}\mu ds = 0,$$

whence we easily obtain

$$\frac{d}{ds}\left(\frac{1}{\mu}\frac{dT}{ds}\right) - \frac{T}{\mu\rho^2} = 0,$$

as above.

The work done by an impulse being equal to the impulse into half the velocity generated [Thomson and Tait, § 308], it follows that the kinetic energy generated in any part of the chain is

$$\tfrac{1}{2}(Tv_s - T'v'_s),$$

T and v_s referring to one end, and T', v_s' to the other end; this may also be written

$$\tfrac{1}{2}(\mu\rho v_s v_\rho - \mu'\rho' v_s' v_\rho').$$

279. *Example* I. As a particular example, suppose a uniform chain to form a semicircle of radius a. Then $\rho = a$, and $s = a\theta$, and (6) becomes

$$\frac{d^2T}{d\theta^2} - T = 0,$$

whose integral is

$$T = A\epsilon^\theta + B\epsilon^{-\theta}.$$

To determine the arbitrary constants, observe that when

$$\theta = 0, \text{ we have } T = T_0,$$

the original impulse; and when $\theta = \pi$, or at the free end of the chain, $T = 0$. Thus we have

$$T_0 = A + B,$$
$$0 = A\epsilon^\pi + B\epsilon^{-\pi}.$$

These give $\quad A = -\dfrac{T_0\epsilon^{-\pi}}{\epsilon^\pi - \epsilon^{-\pi}}, \quad B = \dfrac{T_0\epsilon^\pi}{\epsilon^\pi - \epsilon^{-\pi}};$

and therefore

$$T = T_0\frac{\epsilon^{(\pi-\theta)} - \epsilon^{-(\pi-\theta)}}{\epsilon^\pi - \epsilon^{-\pi}}.$$

The initial velocity at any point can now be easily determined.

280. *Example* II. Suppose it be required that the tension at each point should be proportional to the distance from the free end of the chain.

Then l being the length, and s denoting the same quantity as before,

$$T = T_0\left(1 - \frac{s}{l}\right) \text{ by hypothesis;}$$

$$\therefore \frac{d^2T}{ds^2} = 0, \text{ or by (4) } \frac{T}{\rho^2} = 0, \text{ or } \rho = \infty,$$

that is, the chain must lie in a straight line, as is otherwise evident.

281. *Example* III. Suppose the chain to form a portion of the logarithmic spiral. In this case $\rho = es$ where e is the cotangent of the angle of the spiral. Hence the equation becomes

$$\frac{d^2T}{ds^2} - \frac{T}{e^2s^2} = 0;$$

or, if we put $$s = ae^{\phi},$$

$$\frac{d^2T}{d\phi^2} - \frac{dT}{d\phi} - \frac{T}{e^2} = 0.$$

This is easily integrated, and thus the problem can be completely solved; it is easily shewn that the direction of motion at any point makes a constant angle with the tangent.

282. The investigation of the motion which takes place *after* the impact is not usually considered under Dynamics of a particle—but it is obvious that from what we have just arrived at we may write down the equations of motion of a string in the form

$$\mu \frac{d^2x}{dt^2} = \frac{d}{ds}\left(T\frac{dx}{ds}\right) + \mu X,$$

with two similar equations; the finite forces X, Y, Z, now coming in as we are no longer dealing with impact.

Or, resolving along the tangent and normal, supposing f_s, f_ρ the tangential and normal accelerations at a point, and

S, N the component tangential and normal impressed forces per unit of mass,

$$\mu f_s = \frac{dT}{ds} + \mu S,$$

$$\mu f_\rho = \frac{T}{\rho} + \mu N,$$

and, as before,
$$\frac{df_s}{ds} = \frac{f_\rho}{\rho},$$

T now denoting the finite tension at any point.

As a particular case, if finite (or impulsive) tensions be applied at any two points of a chain of variable density hanging in a given curve at rest under gravity, the tensions being proportional to the tensions in the chain when at rest, the chain will move, as if rigid, vertically.

283. When the string is practically inextensible, and if the tension be great compared with the amount of the external forces; the disturbance being small we may write x for s if we take the undisturbed direction of the string as axis of x.

The equations of transverse vibration become

$$\frac{d^2y}{dt^2} = \frac{T}{\mu} \cdot \frac{d^2y}{dx^2}, \quad \frac{d^2z}{dt^2} = \frac{T}{\mu} \cdot \frac{d^2z}{dx^2},$$

where T is to be regarded as a constant.

The student is particularly to observe that we have now been led to partial differential equations; in fact we have but two equations to represent, for all time, the motion of every point of the string, however the motion of one point may differ from that of another.

The solution is of course of the form

$$y \text{ or } z = \phi(x - at) + \psi(x + at),$$

where $a^2 = \dfrac{T}{\mu}$, ϕ and ψ denoting arbitrary functions.

284. The only other case we shall consider is that of a continuous series of indefinitely small impacts, whose effect is comparable with that of a finite force. The obvious method of considering such a problem is to estimate *separately* the changes in the velocity produced by the finite forces, and by the impacts, in the same indefinitely small time δt, and compound these for the actual effect on the motion in that period.

Another way is to equate the rate of increase of momentum per unit of time to the impressed force.

A mass, under no forces, moves through a uniform cloud of little particles which are at rest. Those it meets adhere to it. Find the motion.

At time t let μ be the mass, and let x denote its position in its line of motion. Then, as there is no loss of momentum, we have

$$\frac{d}{dt}(\mu\dot{x}) = 0.$$

But if M be the original mass, μ_0 the mass of the particles picked up in unit of length, obviously

$$\mu = M + \mu_0 x.$$

Substitute and integrate, supposing $x = 0$, $\dot{x} = V$, when $t = 0$, and we get

$$(M + \mu_0 x)\,\dot{x} = MV,$$

from which x can be easily found.

It is interesting to observe that we have

$$\ddot{x} = -\frac{\mu_0 M^2 V^2}{(M + \mu_0 x)^3},$$

so that the mass moves as if acted on by an attraction $\propto \dfrac{1}{D^3}$ towards a point in its line of motion.

If we take account of the increase of length of the mass in consequence of the deposition of particles on its forward

end, it is obvious that we must write

$$M + \mu_0 (x + \xi)$$

for the mass at time t, where ξ is the increase of length due to the increase of mass. But ξ is obviously proportional directly to the accession of matter, i.e. to $x + \xi$. Hence ξ bears a constant ratio to x; and the only result of this refinement in the solution of the problem is that μ_0 (still a constant) is greater than before: that is, the centre of attraction $\propto \frac{1}{D^2}$ is at a smaller distance behind the origin.

This problem obviously leads to the same result as the following:

A cannon-ball attached to one end of a chain, which is coiled up on a smooth horizontal plane, is projected along the plane. Determine its motion.

285. *A spherical rain-drop, descending under gravity, receives continually by precipitation of vapour an accession of mass proportional to its surface; a being its radius when it begins to descend, and r its radius after the interval t, shew that its velocity is given by the equation*

$$\mathrm{v} = \frac{\mathrm{gt}}{4}\left(1 + \frac{\mathrm{a}}{\mathrm{r}} + \frac{\mathrm{a}^2}{\mathrm{r}^2} + \frac{\mathrm{a}^3}{\mathrm{r}^3}\right),$$

the resistance of the air being left out of account. (Challis, *Smith's Prize Examination,* 1853.)

Let e be the thickness of the shell of fluid deposited in unit of time. Then evidently

$$r = a + et \text{ ....} (1).$$

Also let $\delta v = \delta_1 v + \delta_2 v$ be the increase of velocity in time δt; the first term due to gravity, the second to the impacts.

Evidently, $\delta_1 v = g\delta t$; and if M be the mass at time t, $\delta (Mv) = 0$ is the condition of the impact.

This gives

$$M\delta_2 v = -v\delta M,$$

or
$$\delta_2 v = -v\frac{4\pi r^2 e\delta t}{\frac{4}{3}\pi r^3} = -\frac{3ev\delta t}{r} = -\frac{3ev\delta t}{a+et}.$$

From these we have

$$\frac{dv}{dt} = g - \frac{3ev}{a+et}.$$

Multiplying by $(a+et)^3$, and transferring the last term to the left-hand side of the equation, it gives by inspection

$$(a+et)^3 v = \frac{g}{4e}(a+et)^4 - \frac{g}{4e}a^4.$$

Hence
$$v = \frac{g}{4e}\left\{(a+et) - \frac{a^4}{(a+et)^3}\right\}.$$

Substituting for e from (1),

$$v = \frac{gt}{4(r-a)}\left(r - \frac{a^4}{r^3}\right)$$

$$= \frac{gt}{4}\left(1 + \frac{a}{r} + \frac{a^2}{r^2} + \frac{a^3}{r^3}\right),$$

as required.

To verify this solution, suppose no moisture to be deposited, then $r = a$, and we have $v = gt$ as it ought to be.

Or, immediately from the dynamical equation

$$\frac{d}{dt}(Mv) = Mg,$$

since
$$M = \frac{4}{3}\pi\rho r^3 = \frac{4}{3}\pi\rho(a+et)^3,$$

$$\frac{d}{dt}[(a+et)^3 v] = (a+et)^3 g \text{ or } \frac{dv}{dt} + \frac{3ev}{a+et} = g.$$

286. *One end,* B, *of a uniform heavy chain hangs over a small smooth pully* A, *and the other is coiled up on a table at* C. *If* B *preponderates, determine the motion.*

The moving force due to gravity is the weight of AB minus that of $AC = \mu g(x-a)$ suppose, a being the length AC, and x the length AB.

Now in an indefinitely small interval δt, this would generate in the portion BAC of the chain an increment of velocity

$$\delta_1 v = \frac{\mu g(x-a)}{\mu(x+a)}\delta t.$$

But the whole uncoiled chain, being in motion at the commencement of the interval δt with velocity v, lifts up a portion of length $v\delta t$ from the table during that interval. Hence, if $\delta_2 v$ be the change of velocity arising from this impact, we have by the condition that no momentum is lost,

i.e. $$V' = \frac{MV}{M+M'},$$

$$v + \delta_2 v = \frac{\mu(x+a)v}{\mu(x+a)+\mu v\delta t},$$

or $$\delta_2 v = -\frac{v^2\delta t}{x+a},$$

quantities of the second and higher orders being omitted.

Hence as $$\frac{\delta v}{\delta t} = \frac{\delta_1 v}{\delta t} + \frac{\delta_2 v}{\delta t},$$

proceeding to the limit we have

$$\frac{dv}{dt} = v\frac{dv}{dx} = \frac{g(x-a)-v^2}{(x+a)};$$

which gives $$(x+a)^2 v\frac{dv}{dx} + v^2(x+a) = g(x^2-a^2).$$

Or, immediately, from the equation of momentum,

$$\frac{d}{dt}\left[\mu(x+a)\frac{dx}{dt}\right] = \mu g(x-a).$$

Multiplying by $(x+a)\dfrac{dx}{dt}$ and integrating, supposing $x=b$ initially,

$$\tfrac{1}{2}(x+a)^2\left(\frac{dx}{dt}\right)^2 = g\int_b^x (x^2-a^2)\,dx$$

$$= \tfrac{1}{3}(x-b)(x^2+bx+b^2-3a^2),$$

and this determines for any given initial circumstances the velocity at any instant. The final integration, for the determination of t in terms of x, requires the use of Elliptic Functions; except when $b=2a$, when the acceleration is constant and equal to $\tfrac{1}{3}g$.

(1) If $b<2a$, then $x^2+bx+b^2-3a^2$ will split up into real factors $(x+\beta)(x+\gamma)$ suppose, and we must put

$$x=\frac{b\mp\beta\sin^2\phi}{\cos^2\phi},$$

to reduce the solution to elliptic functions.

(2) If $b>2a$, then $x^2+bx+b^2-3a^2$ is of the form

$$(x\mp\tfrac{1}{2}b)^2+n^2,$$

and we must put

$$x=b\mp c\tan^2\tfrac{1}{2}\phi,$$

where $$c^2=\tfrac{3}{4}b^2\mp n^2.$$

287. If we desire the change produced in the form and position of an orbit by a slight change made in the velocity or direction of motion, &c. at some particular point, we must express separately each of the elements of the orbit in terms of the quantity to be changed; then taking the differentials of both sides, we have the required changes of value.

Thus, we have generally in an elliptic orbit

$$\tfrac{1}{2}v^2=\frac{\mu}{r}-\frac{\mu}{2a}. \quad \S\ 151\ (9).$$

At the end of the major axis farthest from the focus this becomes

$$V^2 = \frac{\mu}{a}\frac{1-e}{1+e}.$$

Now if at this point V be made $V+\delta V$, without change of direction, we have the condition that in the new orbit $a(1+e)$ shall have the same value as in the old; since this will still be the apsidal distance.

Hence

$$\delta(V^2) = \delta\left(\frac{\mu}{a}\frac{1-e}{1+e}\right),$$

and

$$\delta\{a(1+e)\} = 0;$$

$$\therefore\ 2V\delta V = -\frac{\mu}{a}\frac{\delta e}{1+e},$$

or

$$\delta e = -2\sqrt{\left\{\frac{a}{\mu}(1-e^2)\right\}}\,\delta V.$$

And

$$\delta a = -\frac{a}{1+e}\,\delta e$$

$$= 2\sqrt{\left(\frac{a^3}{\mu}\frac{1-e}{1+e}\right)}\,\delta V,$$

which determine the increase of the major axis and diminution of the excentricity; and the same method is applicable to more complicated cases.

Again, in the case of a parabolic orbit, as in Chap. IV., it is easy to see that a change in the magnitude of the velocity shifts the focus in the line joining it with the point of projection through a distance

$$\frac{V\delta V}{g},$$

raises the directrix through an equal distance, and increases the latus rectum by

$$\frac{4V\delta V}{g}\cos^2\alpha,$$

where α is the inclination of the path to the horizon at the instant of the impact.

If the *direction* of motion only be changed, the directrix is unaltered, the focus moves in a direction perpendicular to the line joining it with the point of projection, and the latus rectum is diminished by the length

$$-\frac{4V^2}{g}\sin\alpha\cos\alpha\,\delta\alpha.$$

In the latter case the new orbit again intersects the old, and the tangents to either at the two points of intersection are at right angles to each other; so long as the displacement $\delta\alpha$ is indefinitely small.

These results may easily be extended by geometrical processes, as in Chap. IV., or deduced by differentiation from the analytical results there given.

EXAMPLES.

(1) If $e=1$, one ball cannot be reduced to rest by direct impact on another equal ball, unless the latter is at rest.

(2) If two balls for which $e=1$ impinge directly with equal velocities, their masses must be as 1 : 3 that one may be reduced to rest.

(3) Shew that if two equal balls impinge directly with velocities $\frac{1+e}{1-e}V$ and $-V$, the former will be reduced to rest.

(4) Shew that the mass of the ball which must be interposed directly between M at rest, and M' moving with a given velocity V, so that M may acquire the greatest velocity, is

$$\sqrt{(MM')},$$

and that that maximum velocity is $\frac{M'V(1+e)^2}{\{\sqrt{M}+\sqrt{M'}\}^2}$.

(5) Suppose $e=1$, and an infinite number of balls to be interposed, shew that the maximum velocity which can thus be given to M, is

$$V\sqrt{\frac{M'}{M}}.$$

[Note that, by the result of the preceding question, the masses must form a geometric series, and the above is easily deduced.]

(6) A number of balls A, B, C, &c. for which e is given, are placed in a line; A is projected with given velocity so as to impinge on B, B then impinges on C, and so on; find the masses of the balls B, C, &c. in order that each of the balls A, B, C, &c. may be reduced to rest by impinging on the next; and find the velocity of the n^{th} ball after its impact with the $(n-1)^{\text{th}}$.

(7) A row of elastic balls hanging by long strings, so that their centres are all in the same straight line, are so placed that each ball is almost touching the next; the ball at one end of the row is drawn aside, and permitted to impinge upon that next it; prove that the whole row will remain stationary, except the ball at the other end, which will fly off and rise to a height equal to that from which the first was allowed to descend; the coefficient of restitution being unity.

(8) A given inelastic body is let fall from a given height on one scale of a balance, and two inelastic bodies are let fall from different heights on the other scale, so that the three impacts take place simultaneously; find the relations between the masses and heights that the balance may remain permanently at rest.

(9) Two equal smooth elastic billiard-balls A and B, are placed at a distance d apart, and a third equal ball C is hit so that it impinges on B after striking A. Shew that the loci of all positions of C, whence it is equally easy to make the cannon, are circles whose centres lie on a straight line through A, inclined to AB at an angle $=\frac{1}{2}\pi+\frac{1}{2}\sin^{-1}\frac{4a}{d}$ where a is the radius of the ball, and $e=1$.

(10) An imperfectly elastic ball is projected from a given point in a horizontal plane, against a smooth vertical wall, in a direction making a given angle with the vertical: find where it strikes the horizontal plane, and prove that the locus of these points, for different vertical planes of projection, is an ellipse.

(11) An imperfectly elastic particle is under the influence of a smooth gravitating sphere. Shew that (excepting special circumstances of projection) it will perpetually describe conic sections: determine also the elements of the orbit described after any number of rebounds.

(12) A particle moving in an ellipse about a focus is impinged upon directly by an equal particle moving in a confocal hyperbola about the same centre of attraction. Investigate the nature of the subsequent motion, the coefficient of restitution being unity.

If the excentricity of the elliptic orbit be e, and that of the hyperbolic orbit $\frac{1}{e}$, shew that the apse-line of the new orbit of the former particle is inclined to the apse-line of its old orbit at an angle

$$\operatorname{cosec}^{-1} \frac{1}{3e} \sqrt{4 + e^2 + 4e^4}.$$

(13) A boy standing on a bridge lets a ball fall on the (horizontal) roof of a railway carriage passing under the bridge at 15 miles an hour. If the modulus of elasticity between the ball and carriage roof be $\frac{4}{7}$, and the coefficient of friction $\frac{1}{2}$, find the least height of the boy's hand from the roof that the ball may again rebound from the same point. If the boy's hand be at a greater height than this, what will happen?

(14) A loaded cannon is suspended from a fixed horizontal axis, and rests with its axis horizontal and perpendicular to the fixed axis, the supporting ropes being equally inclined to the vertical; if v be the initial velocity of the ball, whose mass is $\frac{1}{n}$th of the mass of the cannon, and h

the distance between the axis of the cannon and the fixed axis of support, shew that when it is fired off, the tension of each rope is immediately changed in the ratio

$$v^2 + n^2 gh : n(n+1)\,gh.$$

If a cannon be supported in a gunboat in the manner described, with its axis in the direction of the boat's length, what would be the effect of firing it off?

(15) Equal particles revolve in opposite directions about the focus in an ellipse of excentricity $\frac{3}{5}$, and impinge at the nearer apse. Find the distances of future impacts, and shew that if p be the original apsidal distance, the particles fall into the centre of attraction after the time

$$\frac{\pi}{14}\frac{(5p)^{\frac{3}{2}}}{\sqrt{(2\mu)}}.$$

(16) A ball is projected in a given direction within a fixed horizontal hoop, so as to go on rebounding from the surface of the hoop; find the limit to which the velocity will approach, and shew that it attains this limit in a finite time.

(17) If an infinite number of elastic particles, $x = 1$, equally distributed through a hollow sphere, be set in motion each with any velocity, shew that the resulting continuous pressure (referred to a unit of area) on the internal surface is equal to two-thirds of the kinetic energy of the particles divided by the volume of the sphere.

(18) If a spherical bomb-shell resting on the ground burst into a very large number of fragments, all of which are projected with the same velocity, v, in directions uniformly distributed in space, and if the fragments all remain at the place where they first strike the ground, shew that, when all have come to rest, the mass of metal sticking in the ground per unit area at a distance r from where the shell lay is

$$\frac{M}{8\sqrt{2}\pi}\frac{g}{v}\cdot\frac{(v^2 + \sqrt{v^4 - r^2g^2})^{\frac{1}{2}} + (v^2 - \sqrt{v^4 - r^2g^2})^{\frac{1}{2}}}{r\,(v^4 - r^2g^2)^{\frac{1}{2}}},$$

where M is the mass of the shell, and r is great compared with its radius.

Explain the result when $r = \frac{v^2}{g}$.

(19) A hollow cylinder is filled with a very large number of perfectly elastic particles moving freely about in all directions and with all velocities, and impinging on each other and the walls of the cylinder. The cylinder is placed on one of the scales of a balance: shew that the weight of the counterpoise must be equal to the weight of the cylinder and of all the particles together.

(20) A cylinder, length h and radius r, is divided into n equal compartments by n screw surfaces, the pitch of the trace of each on the cylinder being a. It rotates on its axis with angular velocity ω, and a stream of particles moving parallel to the axis with velocities evenly distributed between 0 and V is incident on one end. Shew that the number of particles which pass through the cylinder in unit of time without striking the screw surfaces

$$= \frac{\pi^2 h\omega^2 r^5 \tan\alpha}{nV\left\{h^2\cot^2\alpha - \left(\frac{\pi}{n}\right)^2 r^2\right\}} \times \text{(no. of particles in unit of volume)};$$

provided $$\omega < \frac{h\cot\alpha - \frac{\pi}{n} r}{hr} . V.$$

(21) If at an extremely great distance from the sun meteorites have been flying about equally in all directions for an infinite time, shew that the kinetic energy destroyed per unit of time by meteorites falling into the sun is

$$\frac{\pi M}{2n^3}(1+n^2)^2 r^2 V_0^3,$$

where M is the mass of the meteorites in unit of vol. at a great distance, r = sun's radius, V_0 = velocity from infinity at the sun's surface, and $\frac{V_0}{n}$ = the mean velocity of the meteorites initially.

If one year be the unit of time and the sun's radius the unit of length, shew that this

$$=\frac{\pi' M}{2n^3}(1+n^2)^2(460)^{\frac{3}{2}};$$

having given $r = 400000$ miles, and the earth's mean distance $= 92000000$ miles. Also, from the fact that one unit of heat is equivalent to 772 foot-pounds, find the quantity of heat received by the sun in one year through the impacts.

(22) A train composed of n smooth parallelepipeds is travelling with velocity u along a straight line. A stream of perfectly elastic particles, each of mass m, is moving with velocity v, perpendicular to the line, and is impinging on the train. Supposing that the particles do not interfere with one another, shew that the train experiences a resistance

$$2hNm\{2(n-1)\,av-(n-2)\,bu\}\,u,$$

provided $u<\frac{2av}{b}\nless\frac{av}{b}$, where $a=$ distance between each parallelepiped, b, $h=$ breadth and height of each, and N is the number of particles in a unit of volume.

Can this be used to explain the fact that a train experiences a greater resistance from a cross wind than a head wind?

(23) A comet in moving from one given point to another, throws off at every instant small portions of its mass which always bear the same ratio n to the mass which remains, If v be the velocity with which each particle is thrown off, α the inclination of its direction to the radius vector, prove that the period t will be diminished by

$$\frac{3nvt}{fa}\{(\phi'-\phi)\sqrt{(ap)}\sin\alpha-(r'-r)\cos\alpha\},$$

ϕ and ϕ' being the excentric anomalies, r and r' the focal distances at the given points, a the mean distance, $2p$ the latus rectum, and f the attraction at distance a.

(24) If a rocket, originally of mass M, throw off every unit of time a mass eM with relative velocity V and if M' be

the mass of the case, &c., shew that it cannot rise at once unless $eV > g$, nor at all unless $\frac{eMV}{M'} > g$. If it do rise at once vertically, shew that its greatest velocity is

$$V \log \frac{M}{M'} - \frac{g}{e}\left(1 - \frac{M'}{M}\right),$$

and the greatest height it reaches

$$\frac{V^2}{2g}\left(\log \frac{M}{M'}\right)^2 + \frac{V}{e}\left(1 - \frac{M'}{M} - \log \frac{M}{M'}\right).$$

(25) Particles $(2n-1)$ in number, connected by inextensible strings, are suspended from two fixed points in a horizontal plane so as to hang symmetrically, their weights being such that the inclination of each string to the one immediately below it is α, which is also the inclination of each of the two lowest strings to the horizon. Find their weights; and shew that if the lowest whose mass is m be struck by a vertical blow P, the horizontal component of the initial velocity of any particle varies inversely as its weight, and the vertical component of the initial velocity of the r^{th} from the lowest is

$$\frac{P \sin \alpha}{2m \cos^2 \alpha}\{(2n - 2r - 1) \sin \alpha + 2 \cos \alpha \cot n\alpha - \sin (2r+1)\alpha\}.$$

(26) A large number of equal particles are attached at equal intervals to a string, and the whole is heaped up near the edge of a smooth table; the particle at one extremity of the string is just over the edge of the table. Shew that U_r the velocity of the system just before the $(r+1)^{\text{th}}$ particle is set in motion is given by the equation

$$U_r^2 = \frac{ga}{3} \cdot \frac{(r+1)(2r+1)}{r}.$$

Calculate the dissipated energy.

(27) A very long row of particles, each of mass m, on a smooth horizontal table are connected, each with two adjacent ones, by similar light elastic stretched strings, each

of natural length c; they receive small longitudinal disturbances, such that each of them proceeds to perform a harmonic vibration: prove that there will be two waves of vibrations, in opposite directions, with the same velocity $a\sqrt{\frac{\lambda}{mc}\frac{n}{\pi}\sin\frac{\pi}{n}}$, where a is the average distance between two successive particles, n the number of intervals between two particles in the same phase, and λ the modulus of elasticity.

(28) A light elastic string of length na and coefficient of elasticity λ is loaded with n particles, each of mass m, ranged at intervals a along it, beginning at one extremity. If it be hung up by the other extremity, prove that the period of its vertical oscillations will be given by

$$T=\pi\sqrt{\frac{am}{\lambda}}\operatorname{cosec}\frac{2r+1}{2n+1}\frac{\pi}{2},$$

when $r=0, 1, 2, \dots n-1$, successively. Hence prove that the periods of vertical oscillation of a heavy elastic string will be given by the formula $T=\frac{4}{2r+1}\sqrt{\frac{lM}{\lambda}}$, where l is the length of the string, M its mass, and r is zero or any positive integer.

(29) A uniform chain hangs vertically from its upper end with the lower end just in contact with an inelastic table; if the chain be allowed to fall, prove that the pressure on the table during the fall of the chain is always equal to three times the weight of the coil upon the table.

If the chain hang with its lower end just in contact with a smooth inclined plane, and be let fall, find the pressure on the plane at any time during the fall.

(30) Snow is spread evenly over a roof. If a mass commences to slide, clearing away a path of uniform breadth as it goes, prove that its acceleration is constant, and equal to one-third that of a mass of snow sliding freely down the roof.

(31) The cable of a ship is led through a hole in the deck at a height b above the cable-tier and runs along the deck a distance a, and out at the hawse-hole, immediately outside of which is the anchor, of mass equal to a length $\frac{1}{2}a + 2b$ of the cable. Prove that if the anchor be let go it will descend with acceleration $\frac{1}{8}g$.

(32) A chain of given length is at rest on a smooth horizontal plane, with one end fastened to a point on the plane, under a repulsion from that point varying as the distance. If the chain be set free, find the initial change of tension at any point, and the subsequent motion of the chain.

If the chain impinge upon a vertical wall perpendicular to its own direction, find the pressure upon the wall at any subsequent time.

(33) Two equal weights W are connected by a string of length $2l$, whose weight per unit of length is w, which passes over a small pulley. The system is put in motion by adding a weight W' at one end. Shew that when either weight has moved through a distance x, the kinetic energy will be greater than if the string were weightless by

$$\frac{1}{2}(l-x)^2 w.$$

(34) A fine string passing over a smooth pulley supports two equal scale-pans; a uniform chain is held by its upper end above one of the scale-pans, its lower end being just above the scale-pan; if the upper end be let go, determine the motion completely, and find, at any time, the pressure on the scale-pan.

(35) A pulley is fixed above a horizontal plane. Over the pulley passes a fine string which has two equal chains fastened to its two ends. In the position of equilibrium a length a of each chain is vertical, the remainder of the chains being coiled up on the table.

If now one chain be drawn down through a distance na,

find the equation of motion, and prove that the system will next come to rest when the upper end of the other string is a distance ma below its mean position where

$$(1-m)\,e^{m}=(1+n)\,e^{-n}.$$

If $n=1$, prove that $m=\frac{1}{2}$ approximately.

(36) A uniform flexible chain of indefinite length, the mass of an unit of length of which is m, lies coiled on the ground, while another portion of the same chain forms a coil on a platform at height h above the ground, the intermediate portion passing round the barrel of a windlass placed above the second coil. An engine, which can do H units of work per unit of time, is employed to wind up the chain from the ground and let it fall into the upper coil. Shew that the velocity of the chain can never exceed the value of v determined from the equation

$$mghv+\tfrac{1}{2}mv^{3}=H.$$

(37) A chain whose density varies as the distance from the end A is coiled up close to the edge of a smooth table and the end A allowed to hang over. Shew that the motion is uniformly accelerated and the tension at the edge of the table varies as the fourth power of the time elapsed since the commencement of motion.

(38) A string of length l hangs over a smooth peg so as to be at rest. One end is ignited, and burns away at a uniform rate v. Shew that the other end will, at the time t, before the whole slips off the peg, be at a depth x below the peg, where x is given by the equation

$$(l-vt)\left(\frac{d^2x}{dt^2}+g\right)+v\,\frac{dx}{dt}-2gx=0,$$

given that the mass of the string per unit of length is unity.

(39) A chain is coiled up on a table and is connected with a weight by a fine thread passing over a smooth pulley: if the law of density of the chain be $m\phi\{x\}$; and the mass

producing motion be ml; then the velocity when a length x has been raised is given by

$$\frac{1}{2}v^2 = g\,\frac{l^2x - \int\left\{\int_0^x \phi(x)\,dx\right\}^2 dx}{\left\{l + \int_0^x \phi(x)\,dx\right\}^2}.$$

(40) A series of particles m_1, m_2, ... connected by inelastic strings are placed on a smooth horizontal plane, so that the strings are sides of an unclosed equiangular polygon, each of whose angles is $\pi - \alpha$, and an impulse is applied to the extreme particle P_1 in direction P_2P_1: prove that

$$\frac{T_r - T_{r-1}\cos\alpha}{m_r} = \frac{T_{r+1}\cos\alpha - T_r}{m_{r+1}},$$

where T_r is the impulsive tension of the r^{th} string.

Deduce the equation $\frac{d^2T}{ds^2} - \frac{1}{\mu}\frac{d\mu}{ds}\frac{dT}{ds} - \frac{T}{\rho^2} = 0$ for the impulsive tension at any point of a chain lying in the form of any curve on a horizontal plane and set in motion by tangential impulses, and if the density of the chain vary as the curvature, deduce from either equation that the impulsive tension at any point is equal to $A\epsilon^{\phi} + B\epsilon^{-\phi}$, where ϕ is the angle which the tangent at the point makes with a fixed line, and A, B are constants.

(41) A uniform chain hangs in equilibrium over two smooth pegs in the same horizontal line; if equal vertical impulses be applied simultaneously to the two free ends, find the impulsive tension at any point, and prove that the initial velocity of the vertex of the catenary is to the velocity which would be imparted to each of the straight pieces of chain, if disjointed from the catenary, as $1 : 1 + \sin\alpha$, where α is the greatest inclination of the catenary to the horizon.

(42) A uniform string is lying in a catenary on a smooth

horizontal plane, and the vertex is suddenly projected towards the directrix with a given velocity; shew that the impulsive tension at any point varies as the ordinate of that point, and that every point of the string starts in the same direction.

(43) If a chain of mass m be in the form of a portion of a catenary cut off by a line perpendicular to its axis, and if tangential impulses each equal to mv be applied simultaneously at its two ends, prove that the whole chain will begin to move with the velocity $2v\sin\phi$, where 2ϕ is the angle between the tangents at the ends.

(44) A chain lies upon a smooth horizontal plane in the form of a portion of a common catenary, the tangents at the ends making angles θ_1, θ_2 with the tangent at the vertex of the catenary. An impulsive tension T_1 is applied at the former extremity; shew that the impulsive tension at a point of the chain where the tangent makes an angle θ with the tangent at the vertex is equal to

$$T_1 \frac{\cos\theta_1}{\cos\theta}\,\frac{\theta-\theta_2}{\theta_1-\theta_2}.$$

(45) A string of infinite length is laid on a smooth table in the form of a portion of one branch of the curve $r^n \sin n\theta = a^n$, so that one extremity of the string is at a finite distance from the origin of polar co-ordinates; to this end a tangential impulse is applied, so that the initial direction of motion of each point of the string and the radius vector to the point are equally inclined to the corresponding tangent. Shew that the impulsive tension at any point $\propto r^{-(n-1)}$ and the density of the string must

$$\propto \frac{(r^{2n}-a^{2n})^{-\frac{2n-1}{2n}}}{r^{3n-1}}.$$

(46) A string of varying density slides in a smooth cycloidal tube whose axis is vertical and vertex downwards. Shew that if the string be let fall from any position in which its whole length is within the tube, its centre of gravity will reach the vertex in the same time.

(47) A straight line describes a right circular cone; find the acceleration of a point moving along the line. A string of given length is enclosed in a smooth straight tube, which is made to revolve uniformly about a vertical axis, so as to describe a right circular cone; determine the motion of the string, and the tension at any point.

(48) If a small velocity $n\frac{\mu e}{h}$ be impressed on a planet, in the direction of the radius vector, shew that

$$\delta e = ne \sin(\theta - \varpi),$$

$$\delta\varpi = -n\cos(\theta - \varpi).$$

Calculate also the changes in e and ϖ produced by a small transverse impulse.

(49) A body is moving in an ellipse about the focus; prove that if the body receive a transversal impulse the apse line will be unaffected if the impulse is

$$\frac{mh}{l}(2 + e\cos\theta),$$

where m is the mass of the body, l the semi-latus rectum of its orbit, h is twice the rate of description of area round the focus, and θ is the true anomaly of the body.

(50) If Q be the central disturbing force on a planet, find by Newton's method the equations

$$\frac{de}{d\theta} = -\frac{Qr^2}{\mu}\cdot\sin(\theta - \varpi),$$

$$e\frac{d\varpi}{d\theta} = \frac{Qr^2}{\mu}\cdot\cos(\theta - \varpi),$$

where θ is the longitude of the planet, ϖ the longitude of the apse, e the excentricity of the instantaneous ellipse, r the distance of the planet from the sun.

(51) A particle revolving about a fixed centre to which it is attracted with intensity inversely as the square of the distance is acted on by a small disturbing force f in the direction of the radius vector: prove that the variations of the major axis, the excentricity and the inclination of the line of apses are determined by the equations

$$\frac{da}{dt} = 2e\left\{\frac{a^3}{\mu(1-e^2)}\right\}^{\frac{1}{2}} f\cos(\theta-\omega),$$

$$\frac{de}{dt} = \left\{\frac{a(1-e^2)}{\mu}\right\}^{\frac{1}{2}} f\sin(\theta-\omega),$$

$$e\frac{d\varpi}{dt} = -\left\{\frac{a(1-e^2)}{\mu}\right\}^{\frac{1}{2}} f\cos(\theta-\omega).$$

(52) The first term of the central disturbing force on the moon is $-m^2r$, where the central force is $\frac{\mu}{r^2}$; shew that the apsidal angle (the orbit being nearly circular) is

$$\pi\left(1+\frac{3}{2}\frac{m^2}{n^2}\right) \text{ nearly,}$$

where $\frac{2\pi}{n}$ is a mean lunar month.

(53) A particle is moving in a circle about a centre of attraction $\propto$ (Dist.)$^{-2}$. The strength of the centre increases slowly and uniformly. Determine the approximate elements of the orbit after a given time.

(54) A particle moves in a focal elliptic orbit in a very rare medium whose resistance is as the square of the velocity; determine the effect of the resistance on the periodic time.

(55) A satellite moves about a spherical planet in the plane of its equator, in a slightly elliptic orbit. Find the motion of the apse due to an uniform mountain ridge at the equator.

(56) If when the earth is at an end of the minor axis of its elliptic orbit, a small meteor were to fall into the sun of mass $\frac{1}{m}$ of the mass of the sun, prove that the year would be diminished by $\frac{2}{m}$ of itself.

Prove also that the apse would regrede through the angle $\frac{1}{m}\sqrt{\frac{1}{e^2}-1}$, where e is the excentricity of the earth's orbit.

(57) Central attraction varying as the distance, the velocity of a particle is increased by $\frac{1}{n}$th when it is at the extremity of one of the equal conjugate diameters of its orbit. Shew that each axis is increased by $\frac{1}{2n}$th, and that the apse regredes through an angle

$$\frac{1}{n}\frac{ab}{a^2-b^2}.$$

(58) At what point of an elliptic orbit, described about the focus, can a small change be made in the direction of motion without altering the position of the apse?

If $\delta\phi$ be this change, shew that (in the supposed case)

$$\delta\phi = \frac{\delta e}{1-e^2}.$$

(59) Shew that, in an elliptic orbit about the focus, if the velocity be increased by $\frac{1}{n}$th when the true anomaly is $\theta-\varpi$, we shall have

$$e\delta\varpi = \pm\frac{r\sin(\theta-\varpi)}{na},$$

according as the particle is moving to or from the nearer apse.

(60) A particle moving about a centre of attraction in the focus, in an ellipse of small excentricity, receives a small impulse perpendicular to its direction of motion at any instant. Find the effect on the position of the apse.

(61) Again, if at the extremity of the minor axis the velocity be increased by $\frac{1}{n}$th, and the direction changed so

that h remains the same, find the alteration in the form and position of the orbit.

$$\delta a = 2\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} \delta V,$$

$$\delta e = \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \left(\frac{1}{e} - e\right) \delta V.$$

(62) A particle describes an elliptic orbit about a centre of attraction of intensity varying as (distance)$^{-2}$. If T be the periodic time and a small disturbing force $\lambda \sin \frac{2\pi}{T} . t$ acts in the direction of the radius vector, calculate the variations in the orbit.

(63) A spherical cloud of small masses, whose mutual attraction is insensible, and whose velocities are very small, is overtaken by the sun so as to be incorporated into the solar system. How will the form of the cloud alter as it pursues its approximately parabolic orbit?

(64) The bob of a simple pendulum of length l is acted on by a horizontal force $= pg \cos nt$, where p is a large number, and ln^2 is large compared with g: shew that the pendulum may oscillate about either of two points distant α from the lowest point with an amplitude β where

$$\cos \alpha = 2\frac{ln^2}{gp^2}, \quad \beta = \frac{2}{p}.$$

CHAPTER X.

MOTION OF TWO OR MORE PARTICLES.

288. HAVING considered in detail the various cases which occur in the motion of a single particle subject to any forces, and whose motion is either free, constrained, or resisted, we proceed to the investigation of some very simple cases in which more particles than one are involved. These cases will divide themselves naturally into two series; first, when the particles are entirely free, and are subject to their mutual attractions as well as to other common impressed forces: and second, when there are in addition constraints; such as when two or more of the particles are connected by inextensible strings, &c. Let us take these in order :—

I. *Free Motion.*

289. An immediate application of the third law of motion shews that if two particles attract each other, they exert each on the other equal and opposite forces, in the direction of the line joining them.

If then m, m', be the masses of the particles, and the attraction between two units of matter at distance D be $\phi'(D)$, the intensity is

$$mm'\phi'(D).$$

290. *A system of free particles is subject only to their mutual attractions; to investigate the motion of the system.*

Let, at time t, x_n, y_n, z_n be the co-ordinates of the particle whose mass is m_n, and let $\phi'(D)$ be the law of attraction. Let ${}_pr_q$ express the distance between the particles m_p and m_q; then we have for the motion of m_1,

$$m_1 \frac{d^2 x_1}{dt^2} = \Sigma \left\{ m_1 m_n \phi'({}_1r_n) \frac{x_n - x_1}{{}_1r_n} \right\} \dots\dots\dots\dots\dots\dots (1),$$

$$m_1 \frac{d^2 y_1}{dt^2} = \Sigma \left\{ m_1 m_n \phi'({}_1r_n) \frac{y_n - y_1}{{}_1r_n} \right\} \dots\dots\dots\dots\dots\dots (2),$$

$$m_1 \frac{d^2 z_1}{dt^2} = \Sigma \left\{ m_1 m_n \phi'({}_1r_n) \frac{z_n - z_1}{{}_1r_n} \right\} \dots\dots\dots\dots\dots\dots (3),$$

with similar equations for each of the others; the summations being taken throughout the system. Before we can make any attempt at a solution of these equations, we must know their number, and the laws of attraction between the several pairs of particles. But some general theorems, independent of these data, may easily be obtained: although not nearly so simply as those in Chap. II.

291. I. Conservation of Momentum. In the expression for $m_p \dfrac{d^2 x_p}{dt^2}$, we have a term

$$m_p m_q \phi'({}_pr_q) \frac{x_q - x_p}{{}_pr_q},$$

and in $m_q \dfrac{d^2 x_q}{dt^2}$ we have

$$m_q m_p \phi'({}_qr_p) \frac{x_p - x_q}{{}_qr_p}.$$

Hence if we add all the equations of the form (1) together the result will be

$$m_1 \frac{d^2 x_1}{dt^2} + m_2 \frac{d^2 x_2}{dt^2} + \dots\dots = 0;$$

$$\text{or } \Sigma \left(m \frac{d^2 x}{dt^2} \right) = 0.$$

Similarly $$\Sigma \left(m \frac{d^2 y}{dt^2} \right) = 0,$$

$$\Sigma \left(m \frac{d^2 z}{dt^2} \right) = 0.$$

Now if $\bar{x}$, $\bar{y}$, $\bar{z}$, be at time t the co-ordinates of the centre of inertia of all the particles, § 58,

$$\bar{x}\,\Sigma m = \Sigma\,(mx),$$
$$\bar{y}\,\Sigma m = \Sigma\,(my),$$
$$\bar{z}\,\Sigma m = \Sigma\,(mz).$$

And the above equations may be written,

$$\left.\begin{array}{l} \dfrac{d^2\bar{x}}{dt^2}\,\Sigma m = 0 \\[2ex] \dfrac{d^2\bar{y}}{dt^2}\,\Sigma m = 0 \\[2ex] \dfrac{d^2\bar{z}}{dt^2}\,\Sigma m = 0 \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} \dfrac{d^2\bar{x}}{dt^2} = 0, \\[2ex] \dfrac{d^2\bar{y}}{dt^2} = 0, \\[2ex] \dfrac{d^2\bar{z}}{dt^2} = 0. \end{array}\right.$$

Whence

$$\left.\begin{array}{l} \dfrac{d\bar{x}}{dt} = a \\[2ex] \dfrac{d\bar{y}}{dt} = b \\[2ex] \dfrac{d\bar{z}}{dt} = c \end{array}\right\}.$$

These equations shew that the velocity of the centre of inertia parallel to each of the co-ordinate axes remains invariable during the motion, that is, that *the centre of inertia of the system remains at rest, or moves with constant velocity in a straight line.* See § 72.

The values of a, b, c, may thus be determined,

$$a = \frac{d\bar{x}}{dt} = \frac{\Sigma\left(m\dfrac{dx}{dt}\right)}{\Sigma m}.$$

Now if the initial velocity of m_1 were resolvable into u_1, v_1, w_1, parallel to the axes respectively, and similarly for m_2, &c.

$$a = \frac{\Sigma\,(mu)}{\Sigma m}, \text{ and so for } b, \text{ \&c.}$$

If forces had acted on the particles, of which the components parallel to the axes on the particle m at (xyz) were mX, mY, mZ; we should find

$$\Sigma m \frac{d^2x}{dt^2} = \Sigma mX, \quad \Sigma m \frac{d^2y}{dt^2} = \Sigma mY, \quad \Sigma m \frac{d^2z}{dt^2} = \Sigma mZ;$$

or, which is the same thing,

$$\frac{d^2\bar{x}}{dt^2} \Sigma m = \Sigma mX, \quad \frac{d^2\bar{y}}{dt^2} \Sigma m = \Sigma mY, \quad \frac{d^2\bar{z}}{dt^2} \Sigma m = \Sigma mZ;$$

proving that the motion of the centre of inertia of the system is the same as that of a particle of mass Σm, acted upon by the forces moved parallel to themselves, at the centre of inertia.

292. II. CONSERVATION OF MOMENT OF MOMENTUM. Again, it is evident that if we multiply in succession equation (1) by y_1, and equation (2) by x_1, and subtract, and take the sum of all such remainders through the system of equations of the forms (1) and (2), we shall have

$$\Sigma \left[m \left(x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2} \right) \right] = 0.$$

Integrating once we have

$$\Sigma \left[m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) \right] = 2A_3,$$

where the left-hand member is the moment of momentum of the system about the axis of z.

Now if in the plane of xy we take ρ, θ, the polar coordinates of the projection of m,

$$x \frac{dy}{dt} - y \frac{dx}{dt} = \rho^2 \frac{d\theta}{dt};$$

therefore $$\Sigma \left(m\rho^2 \frac{d\theta}{dt} \right) = 2A_3.$$

Now if a_z be the area swept out by the radius vector ρ on the plane of xy,

$$\tfrac{1}{2}\rho^2 \frac{d\theta}{dt} = \frac{da_z}{dt},$$

and our equation integrated gives

$$\Sigma\,(ma_z) = A_3 t,$$

no constant being necessary if we agree to reckon a_z from the position of ρ at time $t = 0$.

This equation shews (since xy is *any* plane) that generally in the motion of a free system of particles, subject only to their mutual attractions, *the moment of momentum about every axis remains constant;* or, as it is commonly but inconveniently stated, *the sum of the products of the mass of each particle of the system, into the area swept out by the radius vector of its projection on any plane, and about any point in that plane, will be proportional to the time.* See § 72.

Take a_x, a_y to represent for the planes yz, xz the same that a_z represents for xy, then

$$\Sigma\,(ma_x) = A_1 t,$$

$$\Sigma\,(ma_y) = A_2 t.$$

The value of this quantity for a plane, the direction-cosines of whose normal are λ, μ, ν, will be

$$(\lambda A_1 + \mu A_2 + \nu A_3)\,t,$$

and will be a maximum if

$$\lambda A_1 + \mu A_2 + \nu A_3 \text{ is so,}$$

subject to the equation of condition

$$\lambda^2 + \mu^2 + \nu^2 = 1.$$

This gives $\lambda = \dfrac{A_1}{\sqrt{(A_1^2 + A_2^2 + A_3^2)}} = \dfrac{A_1}{A}$ suppose,

with similar values for μ and ν;

and the value of the product for the plane so found is evidently

$$At.$$

Hence, we see also, that, as indeed is evident from the simple statement above, *the axis about which the moment of momentum is greatest remains parallel to itself*, or, as it is usually put, *the plane for which the sum of the products of the masses of the particles into the sectorial areas described by the radii vectores of their projections is a maximum, is a fixed plane or parallel to a fixed plane during the motion.* It has been called on this account the *Invariable Plane.*

If external forces had acted on the system, we should have found

$$\Sigma m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)=\Sigma m\,(yZ-zY),$$

$$\Sigma m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right)=\Sigma m\,(zX-xZ),$$

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\Sigma m\,(xY-yX).$$

293. III. CONSERVATION OF ENERGY. Multiply

$$(1) \text{ by } \frac{dx_1}{dt},\quad (2) \text{ by } \frac{dy_1}{dt},\quad (3) \text{ by } \frac{dz_1}{dt};$$

and, treating similarly all the other equations, add them all together.

Let us consider the result as regards the term on the right-hand side involving the product m_pm_q.

Written at length it is

$$\frac{m_pm_q\phi'({}_pr_q)}{{}_pr_q}\left\{(x_q-x_p)\frac{dx_p}{dt}+(x_p-x_q)\frac{dx_q}{dt}\right.$$
$$\left.+\text{ similar terms in } y \text{ and } z\right\};$$

and the portion in brackets is equal to

$$-\left\{(x_q-x_p)\frac{d}{dt}(x_q-x_p)+\text{similar terms in } y, z\right\};$$

or,
$$-{}_pr_q\frac{d}{dt}({}_pr_q);$$

hence
$$\Sigma\left\{m\left(\frac{dx}{dt}\frac{d^2x}{dt^2}+\frac{dy}{dt}\frac{d^2y}{dt^2}+\frac{dz}{dt}\frac{d^2z}{dt^2}\right)\right\}$$
$$+\Sigma\left\{m_p m_q \phi'({}_pr_q)\frac{d}{dt}({}_pr_q)\right\}=0;$$

therefore, on integration,

$$\frac{1}{2}\Sigma(mv^2)+\Sigma\{m_p m_q \phi({}_pr_q)\}=H.$$

We see therefore that—*the change in the Kinetic Energy of the system in any time depends only on the relative distances of the particles at the beginning and end of that time*, § 78.

294. So far for the case of several particles. The simplest examples will of course be found in the case of two particles only, and to such we will confine our attention; as, when three or more are involved, the problem does not admit of exact solution, and in the two most important applications which have been made of it, namely to the Lunar and Planetary Theories, it is found that a distinct method of approximation is required for each.

Since the acceleration of the centre of inertia is zero, it follows that the motion of each particle with reference to that point is the same as if the latter were at rest. Also, if we apply to each particle of the system an acceleration equal and opposite to that of any one of them, the latter will be reduced to rest, and the relative motion of the others about it will be unchanged. Hence, if there are only two, we see that the relative motion of one about the other will be the same as if the sum of the masses were substituted for the latter.

295. *Two particles, moving initially with given velocities in the same straight line, are subject only to their mutual attraction which is inversely as the square of the distance; to determine the motion.*

The motion will evidently be confined to the straight line. Let m, m' be the masses of the particles estimated on the hypothesis that one unit of mass exerts unit of force on

another unit at unit of distance; x, x' their distances at any time t from a fixed point in the line of motion, then

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \frac{mm'}{(x'-x)^2} \\ m'\frac{d^2x'}{dt^2} &= -\frac{mm'}{(x'-x)^2} \end{aligned}\right\} \qquad \text{..................(1).}$$

Hence, if $\bar{x}$ be the co-ordinate of the centre of inertia at time t,

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = (m+m')\frac{d^2\bar{x}}{dt^2} = 0,$$

$$m\frac{dx}{dt} + m'\frac{dx'}{dt} = (m+m')\frac{d\bar{x}}{dt} = C$$

$$= mV + m'V',$$

if V and V' be the initial velocities; hence the momentum is constant.

Integrating again,

$$mx + m'x' = (m+m')\bar{x} = (mV + m'V')\,t + C'$$

$$= (mV + m'V')\,t + ma + m'a' \text{..........(2)},$$

if a, a' denote the initial positions of the particles.

Again, from equations (1),

$$\frac{d^2(x'-x)}{dt^2} = -\frac{m+m'}{(x'-x)^2},$$

from which, by multiplying by m or m', we see that the relative motion is the same as if the one particle moved to the sum of the masses collected at the other, the position of that other being considered fixed.

Integrating once, we have

$$\frac{1}{2}\left\{\frac{d(x'-x)}{dt}\right\}^2 = C + \frac{m+m'}{x'-x}.$$

At $t=0$, this is

$$\frac{1}{2}(V'-V)^2 = C + \frac{m+m'}{a'-a};$$

and, eliminating C,

$$\frac{1}{2}\left\{\frac{d(x'-x)}{dt}\right\}^2 = \frac{1}{2}(V'-V)^2 + (m+m')\left\{\frac{1}{x'-x} - \frac{1}{a'-a}\right\} \ldots (3).$$

This is of the form

$$\left(\frac{d\omega}{dt}\right)^2 = \frac{A}{\omega} \pm B;$$

therefore $$t = \int \frac{\sqrt{\omega}\,d\omega}{\sqrt{(A \pm B\omega)}},$$

which may be integrated by putting $\omega = y^2$. The integral will be circular or logarithmic according as B is negative or positive. Thus we have $x'-x$ in terms of t, and as we also know $mx + m'x'$ by (2), the motion is completely determined.

If at the instant of projection

$$\frac{1}{2}(V-V')^2 = \frac{(m+m')}{a'-a},$$

the formula (3) becomes

$$\sqrt{(x'-x)}\,\frac{d(x'-x)}{dt} = \pm\sqrt{\{2(m+m')\}},$$

$$\frac{2}{3}(x'-x)^{\frac{3}{2}} = C \pm \sqrt{\{2(m+m')\}}\,t,$$

$$\frac{2}{3}(a'-a)^{\frac{3}{2}} = C,$$

and the motion is completely determined.

296. There is another method of treating this problem. Suppose that, instead of determining the relative motion of the particles, we consider that of each relatively to the common centre of inertia. The distance of m from the centre of inertia is

$$\bar{x} - x = \frac{mx + m'x'}{m+m'} - x = \frac{m'(x'-x)}{m+m'};$$

and we easily find from (1),

$$m\left(\frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2}\right) = -\frac{mm'}{(x'-x)^2} - \frac{m^2}{(x'-x)^2}.$$

Hence, for the relative motion of m and the centre of inertia,

$$m\frac{d^2(\bar{x}-x)}{dt^2}=-\frac{mm'}{(x'-x)^2}$$

$$=-\frac{mm'^3}{(m+m')^2(\bar{x}-x)^2};$$

whence $\bar{x}-x$ may be determined, in finite circular or logarithmic terms, as before.

297. *Two particles, anyhow projected, are acted on solely by their mutual attraction; to shew that the line joining them is always parallel to a fixed plane.* [This is obvious from § 26.]

If m and m' be the particles, x, y, z, x', y', z', their co-ordinates at time t, r their distance, and P the mutual attraction, we have the following equations,

$$m\frac{d^2x}{dt^2}=P\frac{x'-x}{r},\quad m'\frac{d^2x'}{dt^2}=P\frac{x-x'}{r},$$

with similar expressions for the other co-ordinates; hence

$$\frac{\dfrac{d^2(x'-x)}{dt^2}}{x'-x}=\frac{\dfrac{d^2(y'-y)}{dt^2}}{y'-y}=\frac{\dfrac{d^2(z'-z)}{dt^2}}{z'-z},$$

and integrating,

$$(x'-x)\frac{d(y'-y)}{dt}-(y'-y)\frac{d(x'-x)}{dt}=C_3,$$

with other two similar equations. Therefore

$$C_3(z'-z)+C_2(y'-y)+C_1(x'-x)=0.$$

Hence the line joining the particles is always parallel to the plane whose direction-cosines are as C_1, C_2, C_3. This corresponds to § 292.

Also it is evident that the motion of the particles with respect to each other in a plane parallel to this is the same as if the plane were at rest (§ 294).

From the preceding propositions the following are evident deductions.

The centre of inertia of the two particles is at rest only when the initial velocities are zero, or when the directions of projection are the same or parallel, and the momenta equal and opposite.

The plane of relative motion will be at rest only when the initial directions lie in one plane.

If the attraction be inversely as the square of the distance, the relative orbits of the particles about each other, and therefore (§ 27) about their centre of inertia, will be conic sections about a focus.

It is needless to pursue this any further, as the preceding results enable us to reduce the problem to cases treated of in former chapters.

II. *Constrained Motion.*

298. Of the constrained motion of particles, we can only take particular examples, but there are some general considerations which deserve attention.

If two particles be connected by an inextensible string, its only effect is to prevent their relative distance becoming greater than its own length. If we introduce an unknown quantity T for the tension of the string, the equations of motion can be written down, and the condition that the distance of the particles is equal to a given quantity will give us an additional equation, enabling us to eliminate, or to find the value of, this unknown tension. If at any time the value of T so found becomes equal to zero, the motion of the particles must be investigated henceforth as if they were free, until the values of their co-ordinates shew that the string will begin to be tended again. In such a case, if their velocities resolved along the line joining them be not equal, an impact will take place, whose effects must be investigated by the methods of Chap. X.

When the particles are connected by a rigid rod without mass, we have an unknown tension or pressure in the direction of the rod; and, to determine it, we have the geometrical condition that the distance between the particles is constant.

If there be more than two particles attached to the rod, it may exert a transverse force; but cases of this kind more properly belong to the Dynamics of a Rigid Body; and we therefore omit all consideration of them.

299. *Two particles, attached to each other by an inextensible string, are projected with given velocities in space; to determine the motion.*

We may without loss of generality consider the distance between the particles at the instant of projection, to be equal to the length of the string. If their velocities are wholly perpendicular to its direction, or if their resolved parts along it are equal and in the same direction, there will be no impact. If not, suppose the masses m and m' to have velocities v and v' parallel to the string at the instant it is stretched. It is evident that the impact will change each of these into $\dfrac{mv + m'v'}{m + m'}$. This then is determinate; so we may now in addition suppose the resolved parts of the velocities along the string equal to each other. Let x, y, z, x', y', z', be at any time the co-ordinates of the particles, then, if a be the length of the string,

$$m\frac{d^2x}{dt^2} = T\frac{x' - x}{a}, \quad m'\frac{d^2x'}{dt^2} = T\frac{x - x'}{a};$$

and so on.

Also, $$(x' - x)^2 + (y' - y)^2 + (z' - z)^2 = a^2,$$

which are seven equations to find T, and the six co-ordinates of m and m'. From the form of the equations, or by treating them as in § 297, we see that the string remains parallel to a fixed plane, that the centre of inertia moves with constant velocity in a straight line, and that the motion of the particles about each other and about the centre of inertia is the same as if that point were at rest. Hence,

the particles revolve with uniform angular velocity, and the tension of the string is constant. From the above equations

$$T = \frac{mm'}{m+m'} \cdot \frac{V^2}{a},$$

where $V = \sqrt{\left[\left\{\frac{d(x'-x)}{dt}\right\}^2 + \left\{\frac{d(y'-y)}{dt}\right\}^2 + \left\{\frac{d(z'-z)}{dt}\right\}^2\right]}$

is the relative velocity. The same result might have been easily obtained by using the last formula in § 144, when we consider that the velocity of m relative to the centre of inertia is $\frac{m'V}{m+m'}$, that the radius of the circle it describes about that point is $\frac{m'a}{m+m'}$, and that T is the tension which maintains it in that circle.

300. *Two particles, connected by an inextensible string which passes over a small smooth pulley, move under gravity; to determine the motion.*

This was partly anticipated in § 298. Let m, m' be the masses, and let x, x' denote their distances from the pulley at time t. Then if T be the tension of the string (the same throughout since the pulley is smooth), we have

$$m\frac{d^2x}{dt^2} = mg - T,$$

$$m'\frac{d^2x'}{dt^2} = m'g - T.$$

But $x + x' =$ length of string $= a$ suppose. Hence supposing $m > m'$,

$$(m+m')\frac{d^2x}{dt^2} = (m-m')g \text{ ....(1).}$$

This equation completely determines the motion. Also, if we eliminate x and x', we have

$$T=\frac{2mm'}{m+m'}g,$$

and it is therefore constant.

This is one of the cases in which theoretical results may be tested by actual experiment with considerable accuracy. And it was this combination, with many delicate precautions against friction, &c. which Atwood made use of for experimental verification of the laws of motion.

We see, for instance, by equation (1), that we may easily keep $m+m'$ constant while $m-m'$ has any value, and thus by measuring the accelerations produced, find whether they are, in the same mass, proportional to the forces producing the motion. Again, keeping $m-m'$ constant, $m+m'$ may be varied at will. Hence by this process the second law of motion may be tested. See § 68. Again if, while the masses are in motion, a portion be suddenly removed from the greater so that they remain equal, (1) shews us that observation will enable us to test the first law of motion.

So far for the motion when vertical. When the particles are equal, but one of them vibrates as a pendulum, the purely mathematical difficulties of the question become much more serious. From the following approximation however (*Proc. R. S. E.* 1881) we obtain a general idea of the nature of the motion.

Let r, θ be the polar co-ordinates of the vibrating mass—then, neglecting powers of θ higher than the second, we have (§ 250)—

$$2\ddot{r}-r\dot{\theta}^2=-g\frac{\theta^2}{2},$$

$$\frac{d}{dt}(r^2\dot{\theta})=-gr\theta.$$

Put $\frac{g}{2}r$ for r, and $\sqrt{2}\theta$ for θ, and we get

$$\ddot{r}-r\dot{\theta}^2=-\theta^2,$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=-2\theta.$$

Transform to rectangular co-ordinates in the plane of motion—x being vertically downwards:—then

$$\ddot{x} = \frac{y^2}{x^2},$$

$$\ddot{y} = -\frac{2y}{x}.$$

This shews that the vertical acceleration of the vibrating particle is very small but *constantly downward.* Hence the energy of the vibratory motion is steadily converted into energy of translation of the masses. It would be interesting to pursue this question to higher degrees of approximation.

When both the equal masses vibrate through small arcs, it is found that the mass whose *angular* range is the greater has downward acceleration with diminishing angular range. Hence it would appear that, if the string be long enough, the entire motion should be periodic. But the working of this question also is left to the reader.

301. Instead of two masses, connected by a string, suppose a uniform chain of length $2a$ hang over the pulley; then if x be the length hanging down on one side at time t, there will be $2a - x$ on the other, and the difference or

$$2(x - a),$$

is the portion whose weight accelerates the motion. Hence, μ being the mass of the chain per unit of length, we have

$$2\mu a \frac{d^2x}{dt^2} = 2\mu g (x - a);$$

which gives $$x - a = A\epsilon^{\sqrt{\frac{g}{a}}t} + B\epsilon^{-\sqrt{\frac{g}{a}}t}.$$

If the chain were initially at rest, a portion $a + b$ being on one side of the pulley,

$$b = A + B,$$

$$0 = A - B;$$

$$\therefore x - a = \frac{b}{2}\left(\epsilon^{\sqrt{\frac{g}{a}}t} + \epsilon^{-\sqrt{\frac{g}{a}}t}\right).$$

This is true until $x = 2a$, that is, till the chain leaves the pulley; the value of t at that instant being t_0, we have

$$\frac{2a}{b} = \epsilon^{\sqrt{\frac{g}{a}}t_0} + \epsilon^{-\sqrt{\frac{g}{a}}t_0};$$

and therefore $t_0 = \sqrt{\dfrac{a}{g}} \log \left\{\dfrac{a}{b} + \sqrt{\left(\dfrac{a^2}{b^2} - 1\right)}\right\}$.

If, for example, $b = \dfrac{3a}{5}$, i.e. if the portions of the chain were initially as 4 : 1,

$$t_0 = \sqrt{\frac{a}{g}} \log_e 3.$$

302. *Two particles, of masses* m *and* m′, *are attached to different points of an inextensible string, one of whose extremities is fixed. If the system be displaced, to determine the motion.*

Take the axes of x and y horizontal, and that of z vertically downwards, the extremity of the string being origin.

Let a, a' be the lengths of the portions of the string, θ, θ' the angles they make with the vertical, ϕ, ϕ' the angles which vertical planes through them at time t make with the plane of xz. Let x, y, z, x', y', z', be the co-ordinates of the particles and T, T' the tensions of the strings.

Then
$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -T\sin\theta\cos\phi + T'\sin\theta'\cos\phi', \\ m\frac{d^2y}{dt^2} &= -T\sin\theta\sin\phi + T'\sin\theta'\sin\phi', \\ m\frac{d^2z}{dt^2} &= mg - T\cos\theta + T'\cos\theta', \end{aligned}\right\}$$

$$\left.\begin{aligned} m'\frac{d^2x'}{dt^2} &= -T'\sin\theta'\cos\phi', \\ m'\frac{d^2y'}{dt^2} &= -T'\sin\theta'\sin\phi', \\ m'\frac{d^2z'}{dt^2} &= m'g - T'\cos\theta'. \end{aligned}\right\}$$

Besides these, we have the six equations for x, y, z, x', y', z' in terms of a, a', θ, ϕ, θ', ϕ'; in all, twelve equations for the determination of the twelve unknown quantities in terms of t.

303. These equations will be much simplified if we consider the displacement to be in one plane, as the motion will evidently be confined to that plane. By this means we at once get rid of y, y', ϕ and ϕ'. A still greater simplification will be obtained by taking in addition the condition that θ and θ' are so small, that their squares and higher powers may be neglected. With these our equations become

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -T\theta + T'\theta', \\ m\frac{d^2z}{dt^2} &= mg - T + T', \end{aligned}\right\}$$

$$\left.\begin{aligned} m'\frac{d^2x'}{dt^2} &= -T'\theta', \\ m'\frac{d^2z'}{dt^2} &= m'g - T'. \end{aligned}\right\}$$

And, to a sufficient approximation,

$$\begin{aligned} x &= a\theta, \\ x' &= a\theta + a'\theta', \\ z &= a, \\ z' &= a + a'. \end{aligned}$$

Hence, $\qquad T' = m'g$, and $T = (m+m')g$,

$$\left.\begin{aligned} ma\frac{d^2\theta}{dt^2} &= -(m+m')\,g\theta + m'g\theta', \\ m'\left(a\frac{d^2\theta}{dt^2} + a'\frac{d^2\theta'}{dt^2}\right) &= -m'g\theta'. \end{aligned}\right\}$$

Introducing an indeterminate multiplier, and adding,

$$(m+\lambda m')\frac{d^2\theta}{dt^2} + \lambda m'\frac{a'}{a}\frac{d^2\theta'}{dt^2} + \frac{g}{a}\{(m+m')\,\theta + m'(\lambda-1)\,\theta'\} = 0.$$

Let λ_1, λ_2 be the roots of the equation

$$\frac{\lambda}{m+\lambda m'}\frac{a'}{a}=\frac{\lambda-1}{m+m'}.$$

Evidently one is positive and the other negative, and the form of the equation shews that for both $m+\lambda m'$ is positive.

Put $$\phi=\theta+\frac{\lambda m'}{m+\lambda m'}\frac{a'}{a}\theta'=\theta+k\theta', \text{ suppose.}$$

Then the above equation gives

$$\frac{d^2\phi}{dt^2}+\frac{g}{a}\frac{m+m'}{m+\lambda m'}\phi=0.$$

By the recent remark the coefficient of ϕ is positive for both values of λ; let its values be n_1^2 and n_2^2, and we have, k_1, ϕ_1, k_2, ϕ_2, being the corresponding values of k and ϕ,

$$\phi_1=\theta+k_1\theta'=\alpha_1\cos(n_1t+\beta_1),$$
$$\phi_2=\theta+k_2\theta'=\alpha_2\cos(n_2t+\beta_2),$$

where α_1, α_2, β_1, β_2, are arbitrary constants.

Hence,

$$\theta=\frac{1}{k_2-k_1}\{k_2\alpha_1\cos(n_1t+\beta_1)-k_1\alpha_2\cos(n_2t+\beta_2)\},$$

$$\theta'=\frac{1}{k_1-k_2}\{\alpha_1\cos(n_1t+\beta_1)-\alpha_2\cos(n_2t+\beta_2)\}.$$

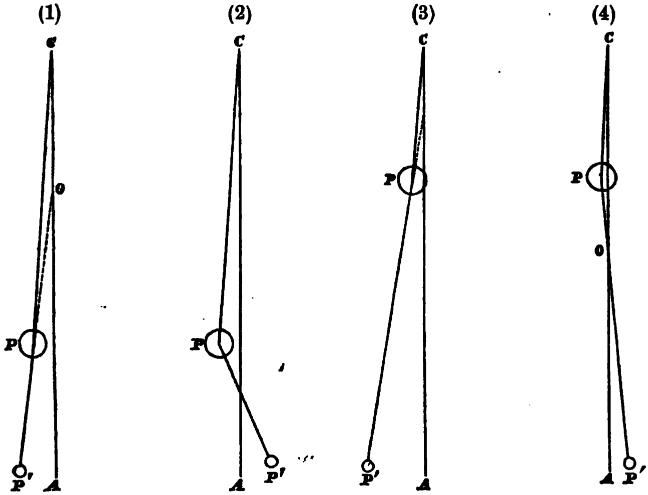
Having given the initial values of θ, θ', $\frac{d\theta}{dt}$ and $\frac{d\theta'}{dt}$, we find α_1, α_2, β_1, β_2, and thus the solution is complete. It may be noticed that the values of θ and θ' may be found at any time by taking the algebraic sum of the corresponding values of the inclinations to the vertical of two pendulums whose times of oscillation are $\frac{2\pi}{n_1}$ and $\frac{2\pi}{n_2}$. Also, if n_1, n_2, be commensurable, the system will in time return to its first position, and the motion will be periodic.

The following discussion of the cases of the simple harmonic motions of the system when m is much greater than m' and the strings are not approximately equal is taken from a paper by Sir W. Thomson, "On the rate of a Clock or Chronometer as influenced by the mode of suspension."

Case I.

The upper string considerably longer than the lower.

Figure 1 represents the first or graver fundamental mode; the period of the upper pendulum CP being made somewhat



graver by the influence of the lower, which in the course of the vibration always exerts a force upon it *from* its middle position.

Figure 2 represents the second or quicker fundamental mode; the vibration of the upper pendulum being in this case excessively small in comparison with that of the lower, and forced by the influence of the latter to a period much smaller than its own would be if undisturbed.

Case II.

The upper string considerably shorter than the lower.

Figure 3 represents the graver mode; the vibration of the upper pendulum through but a very small arc in com-

parison with that of the lower, being augmented by the influence of the lower, which in the course of the vibrations exerts a force upon it always *from* its middle position.

Figure 4 represents the quicker mode; the vibrations of the upper pendulum being made somewhat faster by the influence of the lower, and the lower being influenced so as to vibrate as if it were shortened to the length OA, which is somewhat less than the length CP.

In each case OA is the length of the simple equivalent pendulum vibrating in the same period as that of the fundamental mode represented.

If P consisted of the frame and work of a spring clock, and PP' were its pendulum, then in Case I. the vibrations which would be maintained by the actions of the escapement would be those represented by figure 2, and the clock would go faster than if its frame were perfectly fixed.

In Case II. the vibrations maintained by the escapement would be those represented by figure 3 and the clock would go somewhat slower than its proper rate.

Case I. could never occur in practice, but may be experimentally illustrated by hanging the works of a clock on a light stiff frame moveable round a horizontal axis.

Case II., figure 3, with CP much shorter in proportion to PP' than shewn in the diagram, represents the actual circumstances of an ordinary pendulum clock, which owing to want of perfect rigidity of the frame, must experience a little of the influence of the pendulum there illustrated, causing the rate of the clock to be somewhat slower than it would be if the support of the pendulum were absolutely fixed.

A very slight modification of the process gives us the result of *small* displacements not in one plane: but the student may easily work out these for himself.

We have here a simple example of the principle of the *Coexistence of Small Oscillations;* but this principle, for its satisfactory treatment, requires in general the use of D'Alembert's Principle; which, though (§ 74) merely a corollary to the Third Law of Motion, and clearly pointed out by

Newton as such, is beyond the professed limits of the present treatise.

304. The examples, which have just been given, may suffice to convey an idea of the mode of applying our methods to any proposed case of motion of two constrained particles. These methods are applicable to more complicated cases, when more particles than two are involved; but nothing would be gained by such a proceeding, as D'Alembert's Principle supplies us with a far simpler mode of investigating the motions of any system of free or connected particles: especially when it is simplified in its application by the beautiful system of *Generalized Co-ordinates* introduced by Lagrange (§ 250). See Thomson and Tait's *Natural Philosophy*, §§ 318, 327.

EXAMPLES.

(1) Prove that the periodic time of two bodies round each other is $\frac{2\pi a^{\frac{3}{2}}}{\sqrt{m+m'}}$, where a is their mean distance and m, m' their masses expressed in astronomical units.

(2) If the sun were broken up into an indefinite number of fragments, uniformly filling the sphere of which the earth's orbit is a great circle, shew that each would revolve in a year.

(3) Supposing the earth's present orbit to be circular, examine the effect on the earth of a sudden annihilation of half the sun's mass.

(4) A thin spherical shell of mass M is driven out symmetrically by an internal explosion. Shew that if, when the shell has a radius a, the outward velocity of each particle be v, the fragments can never be collected by their mutual attraction unless

$$v^2 < \frac{M}{a}.$$

(5) Two equal particles are initially at rest in two smooth tubes at right angles to each other. Shew that whatever be their positions, and whatever their law of attraction, they will reach the intersection of the tubes together.

(6) In last question suppose the original distances from the intersection of the tubes to be a, b, and the attraction as the square of the distance inversely, find the future paths if at any instant the constraint is removed.

Solve the same question, supposing the attraction to vary inversely as the cube of the distance.

(7) A shell is describing an elliptic orbit under an attraction tending to the centre. Prove that, if it explodes at any point of its orbit, all the pieces will meet again at the same moment; and that, after half the interval between the explosion and the collision, all the pieces will be moving with equal velocities in parallel directions.

(8) A number of equal particles, attracting each other directly as the distance, are constrained to move in parallel tubes; if the positions of the particles be given at the commencement of the motion, determine the subsequent motion of each; and shew that the particles will oscillate symmetrically with respect to the plane perpendicular to the tubes which passed through their centre of inertia at the commencement of the motion.

(9) Two equal masses M, are connected by a string which passes through a hole in a smooth horizontal plane. One of them hanging vertically, shew that the other describes on the plane a curve whose differential equation is

$$\frac{d^2u}{d\theta^2}+\frac{u}{2}-\frac{g}{2h^2u^2}=0,$$

and that the tension of the string is

$$M\frac{g+h^2u^3}{2}.$$

(10) Two given monkeys cling to a rope, which hangs over the common summit of two given inclined planes; one monkey remains stationary: find the acceleration of the other monkey.

(11) Two equal balls repelling each other with intensity $\propto \frac{1}{D^2}$ hang from the same point by strings of length l. Shew that if when in equilibrium, the strings making an angle 2α with each other, they be approximated by equal small arcs, the time of an oscillation is the same as that of a pendulum whose length is

$$\frac{l \cos \alpha}{1 + 2 \cos^2 \alpha}.$$

(12) One of two equal particles connected by an inelastic string moves in a straight groove. The other is projected parallel to the groove, the string being stretched; determine the motion, and shew that the greatest tension is four times the least.

(13) Two particles connected by a rigid rod move on a vertical circle. Determine the motion, and find the time of oscillation about the position of stable equilibrium.

(14) Two particles P and Q are connected by a rigid rod. P is constrained to move in a smooth horizontal groove. If the particles be initially at rest, PQ making a given angle with the groove in a vertical plane through it, find the velocity of Q when it reaches the groove, and shew that Q's path in the vertical plane is an ellipse.

(15) A particle of mass m has attached to it two equal masses m' by means of strings passing over pulleys in the same horizontal plane, and is initially at rest halfway between them. Determine the motion. Shew that if the distance between the pulleys be $2a$, the velocity of m will be zero when it has fallen through a distance

$$\frac{4mm'a}{4m'^2 - m^2}.$$

(16) Two masses M, M' are connected by a string which passes over a smooth peg. To M' is attached a string which supports a mass m such that $M' + m = M$, and m is displaced through an angle α. Investigate the motion, supposing m so small that the horizontal motion of M' may be neglected. Shew that the string $M'm$ will be vertical after the time

$$\left(\frac{\lambda}{g}\right)^{\frac{1}{2}} \int_0^{\alpha} \left(\frac{1 - \frac{m}{2M}\sin^2\theta}{\cos\theta - \cos\alpha}\right)^{\frac{1}{2}} d\theta,$$

where λ is the length of $M'm$.

(17) Two equal masses are attached each at 1 foot from the ends of a string 3 feet long which are fixed 2 feet apart in a horizontal line. Compare the time of vibration in the various degrees of freedom of the system.

(18) A string $ABCD$, divided into three equal parts at B and C, has two equal weights attached to it, at B and C, and the ends A and D are fastened to two fixed points in the same horizontal plane, the distance AD being two-thirds of the length of the string.

Find the tension of the different portions of the string when there is equilibrium, and, if the string CD be cut through, find the initial changes of tension of the other portions of the string, and the direction and magnitude of the initial acceleration of the weight at C.

(19) The point of suspension of a simple pendulum moves uniformly in a circle in the plane of oscillation of the pendulum, find the equations of motion of the pendulum, and solve them in the case where the radius of the circular arc is very small.

(20) A fine string passes over two smooth pegs in the same horizontal plane and carries three equal weights, one at each end and one capable of sliding on the portion of string between the pegs. If the system be slightly disturbed vertically from its position of equilibrium, find the time of a small oscillation.

(21) A particle of mass M is placed near the centre of a smooth circular horizontal table of radius a; strings are attached to the particle, and pass over n smooth pullies which are placed at equal intervals round the circumference of the circle; to the other end of each of these strings a particle of mass M is attached. Shew that the time of a small oscillation of the system is $2\pi\left(\frac{2+n}{n}\cdot\frac{a}{g}\right)^{\frac{1}{2}}$.

(22) Two particles are attached together by a fine thread: the one is oscillating on the lower part of a vertical circle, the other below the circle and in its plane: if the motions be small, shew that the motion of each particle is compounded of two independent oscillations, the sum of the squares of the periods of which is approximately equal to $\frac{\pi^2 c}{g}$, where c is equal to the sum of the lengths of the radius and the thread.

(23) In a compound pendulum consisting of masses m, m' attached to strings of length l, l', in which of course the most general small motion in one plane consists of two harmonic vibrations superposed, if the upper mass m be very large compared with the under mass m', it is clear that one of the two periodic times (that corresponding to the mode of vibration in which m is nearly at rest) must be very nearly the same as in a simple pendulum of length l', and the other very nearly the same as in a simple pendulum of length l. By a continuous variation of l', the former may be made to pass continuously from less to greater than the latter, and therefore for some value of l' nearly equal to l the two must be equal. But when a system is in stable equilibrium (as is clearly the case here), the equation the roots of which give the times of vibration cannot have equal roots, for that would imply the transitional condition between stable and unstable. Point out precisely the fallacy which leads to the above contradiction.

(24) A string of length na has attached to it at equal intervals n equal particles, and the whole is suspended so as to hang vertically from one end: if the system be slightly dis-

turbed the whole motion will be represented by n simple harmonics whose periods are of the form $\dfrac{2\pi}{\lambda}$, λ being given by the equation

$$0 = \begin{vmatrix} n\left(1-\frac{\omega^2}{\lambda^2}\right), & (n-1), & (n-2), & (n-3), & (n-4),\ldots 2, & 1 \\ (n-1), & (n-1)\left(1-\frac{\omega^2}{\lambda^2}\right), & (n-2), & (n-3), & (n-4),\ldots 2, & 1 \\ (n-2), & (n-2), & (n-2)\left(1-\frac{\omega^2}{\lambda^2}\right), & (n-3), & (n-4),\ldots 2, & 1 \\ (n-3), & (n-3), & (n-3), & (n-3)\left(1-\frac{\omega^2}{\lambda^2}\right), & (n-4),\ldots 2, & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2, & 2, & 2, & 2, & 2\left(1-\frac{\omega^2}{\lambda^2}\right), & 1 \\ 1, & 1, & 1, & 1, & 1 & \left(1-\frac{\omega^2}{\lambda^2}\right) \end{vmatrix},$$

where $\omega^2 = g$.

(25) A spider hangs from the ceiling by an elastic thread whose modulus of elasticity is equal to his weight. Shew that it is possible for him to climb to the ceiling up the thread by the expenditure of $\frac{3}{4}$ of the amount of work required to climb to the same height up an inelastic string, and describe fully the precautions he must take in order to do so.

If the thread be making very small longitudinal oscillations while the spider crawls up very slowly, shew that the time of an oscillation will vary as the square root of the distance of the spider from the ceiling.

(26) Two given masses are connected by an elastic string, and projected so as to whirl round; find the time of a small oscillation in the length of the string.

Give a numerical result, supposing the masses to be 1 lb. and 2 lbs. respectively, and the natural length of the string to be one yard, and supposing that it stretches $\frac{1}{10}$th inch for a tension of 1 lb. weight.

(27) Two particles, connected by an elastic string, are projected in any manner. Shew that in the relative orbit

$$\frac{1}{p^2} = Ar^2 + Br + C.$$

(28) Two particles connected by an elastic string initially unstretched, are projected at right angles to it so that their centre of gravity remains at rest, and their relative velocity is that of a particle falling under gravity through the length of the string. The string is of length a, and would be stretched to a length $2a$ by the harmonic mean of the weights of the particles. Shew that the path of one particle relatively to the other is given by

$$\theta = \int \frac{a^2 dr}{r\sqrt{(r^2 - a^2)\, a^2 - r^2 (r-a)^2}}.$$

Prove also that the string can never become slack.

(29) Two equal particles can slide on opposite horizontal generators of a circular cylinder, and are connected by a stretched elastic string without mass. If the particles are slightly and equally disturbed from their position of equilibrium in opposite directions, find the time of an oscillation. If the unstretched length of string be given, find what must be the radius of the cylinder in order that the time of oscillation may be as small as possible.

(30) A small ring is free to slide along the arc of an ellipse which is placed with its major axis vertical; the ring is supported by an elastic string without weight fastened at the upper focus of the ellipse, and such that its original length was equal to the semi-latus rectum of the ellipse, and of such elasticity that the given ring would, by its weight, stretch it to be equal to the semi-minor axis in length. The ellipse revolves with an angular velocity such that the ring just lies at the extremity of the minor axis. Find the angular velocity, and find the time of a small oscillation of the ring when slightly disturbed along the arc of the revolving ellipse.

(31) A circular elastic band is placed round a wheel the circumference of which is double the natural length of the band; if the wheel be made to revolve with a constant angular velocity, find the pressure of the band on the wheel.

(32) A mass M of fluid is running round a circular channel of radius a with velocity u; another equal mass of fluid is running round a channel of radius b with velocity v; the radius of the one channel is made to increase and the other to diminish till each has the original value of the other: shew that the work required to produce the change is

$$\tfrac{1}{2}\left(\frac{v^2}{a^2} - \frac{u^2}{b^2}\right)(b^2 - a^2)\,M.$$

Hence shew that the motion of a fluid in a circular whirlpool will be stable or unstable according as the areas described by particles in equal times increase or diminish from centre to circumference.

GENERAL EXAMPLES.

(1) A spiral spring is stretched an inch by each additional pound appended to its lower end; find the greatest velocity which will be acquired by 20 lbs. appended to the unstretched spring and allowed to fall.

Also find how far the mass will fall, and the time of a complete oscillation.

(2) Find the form of the hodograph, and the law of its description, for any point of one circular disc rolling uniformly on another. Hence, find the force under which a free particle will describe an epitrochoid, as it is described by a point of the uniformly rolling disc.

(3) The motion of a point P is determined as follows by its position relative to two fixed points A and B. The velocity of P is made up of $\frac{1}{AP}$ *towards* A, and $\frac{1}{PB}$ *from* B. Shew that P describes a circle passing through A and B, and that its velocity at any point is inversely as $AP \, . \, PB$.

If its velocity in any position be the same in magnitude as before, but turned through a right angle in the plane APB, shew that the path is still a circle.

(4) Determine the (unresisted) motion of a body projected vertically at a given point of the earth's surface with a velocity of 7 miles per second.

(5) Apply the principle of varying action to the determination of the (unresisted) motion of a projectile.

(6) Shew that the *action* and *time*, in any arc of the ordinary brachistochrone commencing at the cusp, are represented by the *area* and *arc* of the corresponding segment of the generating circle.

(7) In the parabolic motion of a projectile, the *action* is represented by the area included between the curve, the directrix, and the two vertical ordinates: and the *time* by the intercept on the directrix.

(8) Given a central orbit, and the law of its description, find the differential equation of a curve such that if tangents be drawn to it from any two points of the orbit, the action shall be represented by the area included by these tangents and the two curves.

(9) Find the component harmonic vibrations of two equal simple pendulums hung up by two points in the same horizontal plane, the bobs of the pendulums attracting with intensity as the inverse square of the distance and of magnitude a small portion of the weight of either pendulum.

(10) The point of suspension of a simple pendulum has a horizontal motion expressed by

$$x = a \cos mt.$$

Find the effect on the motion of the pendulum, especially when

$$m^2 = \frac{g}{l},$$

or nearly so, l being the length of the pendulum.

(11) Determine the most general (small) motion of a heavy particle attached at a given point to a stretched elastic string. Shew that it will vibrate with equal rapidity in all directions of displacement, however much the string be stretched, provided the particle be placed at a distance from one end equal to half the length of the unstretched string.

(12) A particle P describes an ellipse freely under the attraction of a second particle G which is constrained to move along the major axis; G, but not P, is attracted to the centre; find the laws of the attractions that PG may be aways the normal at P.

If it were conceivable that P should repel G with the same intensity that G attracts P, a certain relation between the

masses of the two particles would render unnecessary any force to the centre.

(13) A particle P describes an ellipse with constant velocity under two equal forces, one directed towards the focus S, and proportional to CD^n, and the other towards the other focus H; CD being the semidiameter conjugate to CP. Shew that $n = -2$.

(14) A particle P is attached to a point Q by a wire without weight, and is acted on by a force whose accelerating effect varies as the distance from a point O to which it tends; prove that, if Q be constrained to move in a circle with the same velocity as a free particle would describe that circle under the action of the force, P will in all cases move uniformly relatively to Q in a plane parallel to a fixed plane. If QO be the length of the wire, shew that, if P be ever at rest, its absolute path will be a straight line.

(15) A number of equal particles are fastened at equal distances a on an inelastic string, and placed in contact in a vertical line; shew that if the lowest be then allowed to fall freely the velocity with which the n^{th} begins to move is equal to

$$\sqrt{ag\frac{(n-1)(2n-1)}{3n}}.$$

(16) Two particles, of masses m and m' respectively, are connected by a light elastic string of length $2a$. The system is then suspended from a smooth pulley and so adjusted that each particle is at a distance a from the pulley. If the system be then left to itself and y, y' be the distances of the particles from their original positions at the time t, then

$$my - m'y' = (m - m')\frac{gt^2}{2}.$$

(17) A particle attached to the end of a string rests upon a smooth horizontal table; the string passes through a small hole in the table through which it is pulled with uniform velocity; prove that if the particle be acted upon by a force

inversely proportional to its distance from the hole, and perpendicular to the string, it will describe if properly projected an equiangular spiral.

(18) A centre of attraction which attracts a particle of mass m with intensity $m\omega \times$ distance, moves on the circumference of a circle of radius a with constant angular velocity ω, and a particle is placed midway between the centre of attraction and the centre of the circle; if r and θ be its polar co-ordinates when the centre of attraction has moved through an arc $a\phi$, prove that

$$r = \frac{a}{2}(1 + \phi^2), \quad \theta + \phi = \tan\theta,$$

the centre of the circle being the pole, and the initial line passing through the initial position of the particle.

(19) Three particles each of mass m are lying on a smooth horizontal table in a straight line joined together by two strings, each of length a. The two outer particles are projected simultaneously with the same velocity v in a direction perpendicular to the strings, prove that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{v^2}{a^2}\frac{1}{2 - \cos 2\theta},$$

where θ is the angle the string joining the middle particle with either of the other two has turned through in any time.

(20) Three equal particles are joined by two equal strings and are placed in one straight line on a smooth table; if the middle one be projected perpendicular to the string with a velocity V, the velocity of the other two when they impinge is $\frac{2}{3}V$.

(21) Two particles are joined by a string which passes through a small ring, the particles are held in the same horizontal line, and the string is tightened and then let go; if ρ, ρ' be the radii of curvature of their paths initially, a, a'

the initial lengths of the portions of the string, m, m' their masses, shew that

$$\frac{m}{\rho} = \frac{m'}{\rho'} \text{ and } \frac{1}{a} + \frac{1}{a'} = \frac{1}{\rho} + \frac{1}{\rho'}.$$

(22) Investigate the equation of motion of a chain constrained to move in a fine tube under given forces.

A uniform chain of length $4a$ is coiled up on a horizontal table at the distance a from one edge of the table, and one end of the chain is then drawn out at right angles to the edge and just over it; the height of the table above the floor being a, investigate completely the motion of the chain.

(23) An elastic string of length a, mass ma, is placed in a tube in the form of an equiangular spiral with one end attached to the pole. The plane of the spiral is horizontal, and the tube is made to revolve with uniform angular velocity ω about a vertical axis through the pole; prove that its length, when in relative equilibrium, is given by the equation

$$l = a\frac{\tan\phi}{\phi},$$

where
$$\phi = a\omega \cos \alpha \sqrt{\frac{m}{\lambda}}.$$

(24) A particle is suspended from a fixed point by an elastic string, and performs small oscillations in a vertical direction; supposing the string uniform in its natural state and of small finite mass, shew that the time of oscillation will be approximately the same as if the string were without weight and the mass of the particle increased by one-third of that of the string.

(25) The resistance of the æther to a planet or comet moving with the velocity V being assumed to be $k\dfrac{dV}{dt}$ and the sun's attraction being $\dfrac{\mu}{r^2}$, obtain the following exact equations:

$$V^2(1+k)-\frac{2\mu}{r}=C,$$

$$r^2\frac{d\theta}{dt}=hV^k.$$

Obtain also the differential equation of the orbit in the form

$$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{(C+2\mu u)^{1-k}}{h^2(1+k)^{1-k}}.$$

(26) A body moves in a plane about a fixed point under given forces. If the areal velocity and the direction of motion of the body at a proposed point be known, find the semi-latus rectum of the elliptic orbit which has a contact of the second order with the real orbit at that point, its focus being at the given fixed point.

Also find the changes produced in an indefinitely small time in the excentricity and in the position of the apse in this elliptic orbit in terms of the corresponding change of the semi-latus rectum.

(27) Prove that the apparent path of a comet on the celestial sphere is concave or convex towards the sun's apparent place according as the comet or the earth is nearer to the sun.

(28) It has been found by comparing theory with observation that the perihelion of Mercury progresses at a rate greater by α than that due to the attraction of known bodies; shew that this increment would be accounted for if the law of force tending to the sun were $\frac{\mu}{r^2}+\frac{\mu'}{r^4}$, and if $\mu'=\alpha\sqrt{\frac{\mu}{c}}$, the orbit being supposed to be nearly a circle and the mean distance to be c.

(29) A comet moving in a parabolic orbit makes a near approach to a planet; point out from general considerations the circumstances under which the orbit of the comet is rendered elliptic or hyperbolic.

(30) A particle moves under a retardation $f(t)$ which brings it to rest in a time a; prove that the distance traversed is

$$\int_0^a tf(t)\,dt.$$

(31) If the velocity of a railway train resisted by friction differ from its mean velocity by a periodic function of the time, determine the least horse power of the engine that will draw the train, and prove that this horse power is greater than what would be required if the velocity were constant.

(32) A particle is in motion within a triangle ABC, and is attracted perpendicularly to the sides with intensities each equal to μ times the perpendicular distance. Shew that the motion consists of two periodic terms of the form

$$P \sin \{t\sqrt{\lambda\mu} + Q\},$$

where $$(\lambda - 1)(\lambda - 2) + 2 \cos A \cos B \cos C = 0.$$

Shew distinctly that the roots of this quadratic are real and positive; examine the case of an equilateral triangle and in that case verify the above result independently.

(33) A particle is attracted perpendicularly towards the faces of a tetrahedron with intensities equal respectively to μ^2 times the perpendicular distances. If the medium resist with intensity $2kv$, then the particle on moving within the tetrahedron will have its motion stable provided the equation in λ

$$\begin{vmatrix} \dfrac{\lambda^2 + k^2}{\mu^2} - 1, & \cos(12), & \cos(13), & \cos(14) \\ \cos(21), & \dfrac{\lambda^2 + k^2}{\mu^2} - 1, & \cos(23), & \cos(24) \\ \cos(31), & \cos(32), & \dfrac{\lambda^2 + k^2}{\mu^2} - 1, & \cos(34) \\ \cos(41), & \cos(42), & \cos(43), & \dfrac{\lambda^2 + k^2}{\mu^2} - 1 \end{vmatrix} = 0,$$

has all its roots real, cos (12) being interpreted to mean the cosine of the angle between two faces which are marked 1, 2 respectively.

(34) A smooth cylinder, whose transverse section is a cycloid generated by a circle of diameter a, is placed with its axis horizontal, the axis of the cycloidal section being vertical and its vertex downwards. A particle is allowed to fall from rest at any point of the surface and is attracted by a perfectly elastic plane perpendicular to the axis of the cylinder, with an intensity varying directly as the distance from the plane, and strength $\frac{2g}{a}$. Shew that the path of the particle will be such that if the cylinder be developed it will develope into successive portions of a parabola.

(35) A vertical wheel rolls on a horizontal plane with the velocity it would acquire by falling through a height equal to half its radius; a particle flies off at the point P; shew that the focus of the parabola described by the particle is the foot of the perpendicular from the lowest point of the wheel upon the radius through P; and that the focal distance of P is a mean proportional between the semi-latus rectum and the radius of the wheel.

(36) From every point of an ellipse particles are projected in the direction of the tangent with velocities such that, when under a central attraction $\frac{\mu}{r^2}$ to one of the foci of the ellipse, they proceed to describe parabolas. Shew that the directrices of these parabolas all touch one or other of two fixed circles whose radii are equal to the major axis of the given ellipse.

(37) Find the circumstances of projection that a particle attracted by an infinite straight line with intensity inversely as the square of the distance may describe a set of complete cycloids.

(38) A particle is revolving on a smooth plane about a centre of attraction, of intensity $\mu \times$ distance, and when

the body arrives at an apse the plane begins to revolve with an angular velocity $\frac{1}{2}\sqrt{3\mu}$ about the apsidal line; shew that the subsequent orbit described on the plane will be a portion of a parabola; and that, when the particle leaves the plane, its velocity will be $\sqrt{3} \times$ velocity at the vertex.

(39) If the component velocities, parallel to two rectangular axes, of a particle moving in a plane be proportional to its distances from two other straight lines in the plane at right angles to each other, its path will be an equiangular spiral or a rectangular hyperbola.

(40) Prove that if a point move in a plane curve with velocity v, and if the direction of its motion make an angle ψ with a fixed line, the rate of change per unit time of the magnitude of v is $\frac{dv}{dt}$ and of direction $v\frac{d\psi}{dt}$.

Deduce the expressions for the accelerations when the position of the point is given by its distance from a fixed point and the angle which that distance makes with a fixed line.

(41) A point is moving on a plane area, which is itself moving in any way in its own plane. Find the accelerations of the point with regard to absolutely fixed axes.

A_1 describes an equiangular spiral with uniform angular velocity about O: A_2 describes an equal spiral with the same angular velocity about A_1, and so on. Prove that A_n describes an equiangular spiral with the same angular velocity relatively to O, and find its size.

(42) Assuming that the moon describes areas proportional to the times of describing them by radii drawn to the centre of the earth, examine the nature of the force which acts on the moon. On the above assumption and taking the orbits of the moon and earth to be circular, shew that the acceleration of the moon in the direction of the tangent to its orbit on the nth day of the (lunar) month is $\frac{g}{1800}\sin\frac{n\pi}{14}$

roughly. Given that the distance of the sun from the earth's centre is 24000 times the earth's radius, and that the mass of the sun is 320000 times that of the earth.

(43) Two particles revolve about a centre of attraction (the law of attraction being as the distance), one in an ellipse of excentricity $\frac{2}{\sqrt{7}}$, and the other in a circle passing through the foci of the ellipse. Shew that the first particle moves within the circular path of the second particle during $\frac{1}{3}$ of its period; and compare the velocities of the two particles at the points common to their orbits.

(44) If the resolved part perpendicular to the radius vector of the velocity of a body revolving in an elliptic orbit about the focus ever be half its whole velocity, shew that the excentricity of the ellipse must be $> \frac{\sqrt{3}}{2}$: and that it is impossible that at the same time the resolved part of its velocity perpendicular to the major axis should be also half its whole velocity.

(45) If a particle be projected from an apse at a distance a from a centre of attraction of which the intensity at distance r is $\mu(r-a)$, obtain the equation for determining the other apsidal distance, and find the velocity of projection in order that it may be $\frac{3a}{2}$.

(46) If the orbit is $p^2(a^{m-2}-r^{m-2})=b^m$, shew that the apsidal angle is $\frac{\pi}{\sqrt{m}}$ nearly.

(47) A particle of mass m is attached to a fixed point by an elastic string of natural length a, whose coefficient of elasticity is m. It is projected with the velocity due to half the length of the string, in a direction perpendicular to the string which is initially unstretched. Prove that the apsidal distances of its orbit are given by

$$r^4-2ar^3+a^4=0.$$

(48) Particles describe confocal ellipses under the attractions tending towards the centre. If at any instant they are all at the ends of the conjugate, or transverse, axes of their orbits, prove that a hyperbola confocal with the ellipses can always be drawn through them all.

(49) A particle is moving in an ellipse about a centre of attraction in the focus, and the centre of attraction is transferred to one end of the latus rectum as the particle passes through the other. Prove that e, e', the excentricities of the old and new orbits, are connected by the relation

$$e'^2 = 1 + 4e^2.$$

(50) A body describes a fixed ellipse under an attraction to the focus, and a second body describes a similar and equal ellipse which revolves in its own plane about its focus which is fixed, while the plane itself moves so as to retain the same inclination to a fixed plane, the bodies being always at corresponding points in the two ellipses; if the angular velocity of the line of intersection of the two planes, and also the angular velocity of the axis of the ellipse with respect to this line, be always proportional to the corresponding angular velocity of the body in the fixed ellipse, find the forces requisite to make the second body move in the manner thus defined.

Also find the elements of the orbit which would be described by the second body, if the forces acting upon it were at any moment replaced by an attraction tending to the focus and equal to the attraction in the fixed plane.

(51) A particle moves under a force whose magnitude is proportional to the distance from the axis of x, and whose direction is always perpendicular to the path of the particle. The particle is projected from the point $x = -a$, $y = a$, parallel to the axis of y, with velocity $a\sqrt{\frac{\mu}{2}}$. Shew that the path described is

$$\sqrt{2a^2 - y^2} + x = \frac{a}{\sqrt{2}} \log \frac{\sqrt{2a^2 - y^2} + a\sqrt{2}}{y(1 + \sqrt{2})}.$$

(52) Investigate the equations of motion of a particle attracted to any number of centres.

A particle can describe a certain orbit under an attraction P to the point S, and it describes the same orbit under an attraction P' to the point S'. Find the necessary conditions that it may describe the same path when acted on both by P and P'.

Two centres attracting inversely as the square of the distance are distant r, r' respectively from a particle moving under their influence; if θ, θ' be the angles r, r' make with the line joining the centres of force, then

$$r^2 r'^2 \frac{d\theta}{dt}\frac{d\theta'}{dt} = a\left(\mu\cos\theta + \mu'\cos\theta' + c\right),$$

μ, μ' being the absolute intensities of, and a the distance between, the centres of force, and c an absolute constant.

(53) If a parabola be described under two forces one constant and parallel to the axis, and the other a repulsion from the focus inversely as the square of the distance, find the time of describing any arc of the parabola.

(54) A particle is under a central attraction

$$\mu\left\{u^3 - l^2u^2\frac{\left(u-\frac{1}{l}\right)^3}{e^2}\right\},$$

and is projected from an apse at a distance $\dfrac{l}{1+e}$ with velocity $\dfrac{1+e}{l}\sqrt{\mu}$, shew that the orbit described has for equation

$$\frac{l}{r} = 1 + e\,\mathrm{cn}\left(\theta, \frac{1}{\sqrt{2}}\right).$$

(55) A body is placed on a rough inclined plane, whose inclination is greater than $\tan^{-1}\mu$, and is connected with an elastic string parallel to the plane and attached to a fixed point. If initially the body be at rest and the string of its natural length, determine the circumstances of the resulting motion.

(56) Particles move each in a system of confocal and co-axial parabolas under a force constant for each particle and tending to the focus; at the beginning of the motion they lie on a straight line passing through the focus: shew that this will always be true if the forces and velocities of projection are proportional to the latera recta.

(57) A particle moves under gravity on a smooth curve in a vertical plane, and after leaving the curve describes a parabola freely, and whatever be the velocity the vertical ordinate of the point where it leaves the curve bears to the vertical ordinate of the highest point attained in the free path the ratio $2 : m+1$; prove that the equation of the curve is $y^m = cx^{m-1}$.

(58) A particle is placed on the surface of a smooth fixed sphere, of radius c, at an angular distance α from its highest point; prove that the latus rectum of the parabola which the particle describes after leaving the sphere is $\frac{16}{27} c \cos^3 \alpha$; and find the range on the tangent plane at the lowest point of the sphere.

(59) A particle is placed very near the vertex of a smooth cycloid, having its axis vertical and vertex upwards; find where the particle runs off the curve, and prove that it falls upon the base of the cycloid at the distance $\left(\frac{\pi}{2} + \sqrt{3}\right) a$ from the centre of the base, a being the radius of the generating circle.

(60) A smooth right circular cylinder is placed with its axis horizontal, and a particle moving with velocity v along the lowest generating line receives a horizontal impulse at right angles to this line and just sufficient to carry it to the highest point of the cylinder. If the particle be prevented from leaving the cylinder, shew that its subsequent path is such that if the cylinder be developed its equation is

$$y = \pi a - 4 \tan^{-1} \epsilon^{-\frac{x}{v}\sqrt{\frac{g}{a}}},$$

and that the highest generating line is an asymptote to the curve.

(61) A hyperbola is placed in a vertical plane with the transverse axis horizontal; prove that when the time of descent down a diameter is least, the conjugate diameter is equal to the distance between the foci.

(62) Find a curve such that the time of descent down all tangents from the point of contact to a given horizontal line is the same.

(63) Prove in an elementary manner that the line of quickest descent, from one curve in a vertical plane to another in the same plane, is such that it bisects the angle between the normal and the vertical at each extremity.

If the two curves are (i) an ellipse of semi-axes a, b, having its major axis ($2a$) vertical, and (ii) a concentric circle of radius $c\,(<b)$, then the length of the normal to the ellipse at one extremity of such a chord, intercepted between the ellipse and the major axis, is

$$\frac{b}{a^2}\{bc+\sqrt{(a^2-b^2)\,(a^2-c^2)}\},$$

and the time of transit of the particle is

$$\frac{2}{a}\sqrt{\frac{\sqrt{a^2-b^2}}{g}\{b\sqrt{a^2-c^2}-c\sqrt{a^2-b^2}\}}.$$

(64) A series of curves $f(x, y, \lambda)=0$ are described in a vertical plane, and the lines of quickest descent are drawn to them from the point (h, k). Shew that the locus of their extremities is found by eliminating λ between

$$(k-y)\left\{\sqrt{\left|\frac{df}{dx}\right|^2+\left|\frac{df}{dy}\right|^2}-\frac{df}{dx}\right\}-\frac{df}{dy}(h-x)=0$$

and $$f(x, y, \lambda)=0.$$

If (h, k) lies on a straight line, the envelope of these curves depends only on the inclination of the line to the vertical.

Ex. In a series of similar and similarly situated concentric ellipses, if the point (h, k) is the common centre, the above locus degenerates into the axis of y and two straight lines equally inclined to it at an angle $\cot^{-1}\sqrt{3-2e^2}$.

(65) Shew that if the time which a particle takes to move from rest at any point of a smooth curve to a fixed point on the curve is constant, the acceleration resolved along the curve must be proportional to the arcual distance from the fixed point. Hence shew that the hypocycloid is isochronous for an attraction varying as the distance from the centre of the fixed circle: and that the time of an oscillation is

$$\frac{2\pi}{a}\sqrt{\frac{(a-b)\,2b}{\mu}},$$

where a and b are the radii of the fixed and rolling circles.

(66) Shew that the parabola $r = a\sec^2\dfrac{\theta}{2}$ is a brachistochrone for a force which varies as $(r\sin\theta)^{-3}$ and acts at right angles to the radius vector, if the particle is properly projected.

(67) Shew that if a particle move from one given point to another under any forces, the time integral of its kinetic energy is stationary for small variations of the path.

Shew that if v be the velocity of the particle at any point the differential equations to its path are

$$\frac{\delta v}{\delta x} - \frac{d}{ds}\left(v\,\frac{dx}{ds}\right) = 0,$$

and two similar equations.

Explain what is meant by equi-actional surfaces, and distinguish clearly between them and equi-potential surfaces.

(68) A particle moves on a curved surface under any forces; if R be the pressure on the surface prove the equation

$$\frac{v^2}{\rho} = P + R,$$

P being the resolved part of the impressed force in the direction of the normal, v the velocity and ρ the radius of curvature of a normal section of the surface through the tangent to the path.

If the surface be one of revolution and the resultant of the impressed forces passes through the axis,

$$\tfrac{1}{2}v^2 = C + \int (Pdr + Zdz);$$

the axis of z being that of revolution, and P and Z the forces in the normal and parallel to the axis respectively.

(69) A smooth hollow ellipsoid rests with an axis vertical. A particle is moving very near to the lowest point of the surface, determine its motion.

(70) A tube revolves round the axis of x with an angular velocity (ω), shew that if the impressed forces be X, Y, parallel and perpendicular to the axis, and the particle be in relative equilibrium at a point at which the radius of curvature is (ρ) and the inclination of the tangent to the vertical (a), then the motion will be stable or unstable according as

$$\omega^2 \lessgtr \frac{\cos a\left(\dfrac{dX}{ds} - \dfrac{Y}{\rho}\right) + \sin a\left(\dfrac{dY}{ds} + \dfrac{X}{\rho}\right)}{\dfrac{y}{\rho}\cos a - \sin^2 a}.$$

(71) A railway train of given mass is travelling due south at a uniform rate along a line which runs due north and south: prove that, the earth being supposed perfectly spherical, the train will exert a pressure on the western metals, the magnitude of which varies as the product of the velocity of the train and the sine of the latitude of its position, and a pressure towards the south, the magnitude of which varies simply as the sine of twice the latitude.

(72) If a very small tangential disturbance, f, act on a particle oscillating in a cycloid, prove that the increase in the arc of semi-vibration is equal to

$$2\sqrt{\frac{a}{g}} \cdot \int f \cos\left(\sqrt{\frac{g}{4a}} \cdot t + \beta\right) dt,$$

the integration extending over the time of a semi-vibration. Also find an expression for the proportionate increase in the time of oscillation.

(73) A particle moves in a resisting medium: shew how to find the resistance that a given curve may be described, the force acting in parallel lines. A particle describes a curve under a constant acceleration which makes a constant angle with the tangent to the path: the motion takes place in a medium resisting as the n^{th} power of the velocity. Shew that the hodograph of the curve described is of the form

$$b^{-n}e^{-n\theta\cot\alpha} = r^{-n} - a^{-n}.$$

(74) A particle is moving under a central attraction and experiences resistance which varies as the square of its velocity. Find a differential equation for its orbit.

If the attraction is $\frac{\mu}{r^2}$, and the resistance kv^2, k being small, shew that at the beginning of motion the velocity is given by

$$\tfrac{1}{2}v^2 = \int \mu\epsilon^{2ka\theta}\, a \sin\theta \,.\, d\theta,$$

where $a \pm \alpha$ are the reciprocals of the maxima and minima values of the radius vector, α being supposed small, and θ the angle described from the beginning of motion.

(75) A rain-drop descending through a damp atmosphere at rest, accumulates moisture so that the radius increases uniformly. If a sudden gust of wind gives it a horizontal velocity, shew that it will proceed to describe a hyperbola one of whose asymptotes is vertical.

(76) A spherical rain-drop descending from rest by the action of gravity receives continual accessions to its mass by depositions of vapour proportional to its surface: the radius of the drop being a at starting, and r after an interval t, the velocity acquired in the same interval being V, shew that

$$V = \frac{gt}{4r^3} \cdot \frac{r^4 - a^4}{r - a},$$

if the resistance of the air be not taken into account.

Solve the same problem, supposing the resistance of the

air to be in a given ratio to the actual acceleration of the drop independently of its size.

(77) If a, b, c, d, e, f are the six elements introduced by the integration of the equations

$$\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}=\frac{dR}{dx},\quad \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}=\frac{dR}{dy},\quad \frac{d^2z}{dt^2}+\frac{\mu z}{r^3}=\frac{dR}{dz},$$

on the hypothesis $R=0$, and if x', y', z' the expressions for

$$\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt},$$

in terms of the time and these elements have the same form whether R be zero or finite, prove that

$$\frac{dR}{da}=[a,\ b]\frac{db}{dt}+[a,\ c]\frac{dc}{dt}+\ldots+[a,\ f]\frac{df}{dt},$$

where

$$[a,\ b]=\frac{dx}{da}\frac{dx'}{db}-\frac{dx}{db}\frac{dx'}{da}+\frac{dy}{da}\frac{dy'}{db}-\frac{dy}{db}\frac{dy'}{da}+\frac{dz}{da}\frac{dz'}{db}-\frac{dz}{db}\frac{dz'}{da}.$$

Next, representing the solution by the system of equations

$$nt+c=u-e\sin u,\qquad n^2a^3=\mu,$$

$$\xi=a\cos u-ae,\qquad n=a\sqrt{1-e^2}\sin\mu,$$

$$x=(\xi\cos\gamma-\eta\sin\gamma)\cos\Omega-(\xi\sin\gamma+\eta\cos\gamma)\cos\iota\sin\Omega,$$

$$y=(\xi\cos\gamma-\eta\sin\gamma)\sin\Omega+(\xi\sin\gamma+\eta\cos\gamma)\cos\iota\cos\Omega,$$

$$z=(\xi\sin\gamma+\eta\cos\gamma)\sin\iota,$$

where u is an auxiliary angle, and $a, c, e, \gamma, \iota, \Omega$ are the six elements, prove that

$$[\iota,\ \Omega]=na^2\sqrt{1-e^2}\sin\iota.$$

State also how u must be dealt with in calculating $[a, c]$.

APPENDIX.

A. *On the integration of the equations of motion about a centre of attraction.*

In general (Chap. V.), the problem of central attraction is solved by considering the equation connecting $u\left(\text{or } \frac{1}{r}\right)$ and θ, and employing the resulting integrated relation between r and θ to find θ in terms of t from the law of equable description of areas. If we try to express r and θ separately, in terms of t, without first determining the form of the orbit, we are led to a host of curious results which may be easily obtained; so easily indeed, that we shall merely notice one or two of them.

From the usual equations for motion about a centre, *i.e.*

$$\frac{d^2x}{dt^2} = -P\frac{x}{r},$$

$$\frac{d^2y}{dt^2} = -P\frac{y}{r},$$

where P is the acceleration due to the central attraction, we get at once

$$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}\left(\frac{dy}{dt}\right)^2 = -\int P dr,$$

and
$$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} = -Pr.$$

Adding, we have immediately,

$$\frac{d}{dt}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \tfrac{1}{2}\frac{d^2(r^2)}{dt^2} = -2\int Pdr - Pr \ldots(1).$$

This, for any assigned form of P in terms of r, will evidently give us r^2 in terms of t.

Now there is a remarkable case in which r^2 can be generally expressed as a rational integral function of t. Suppose

$$-2\int Pdr - Pr = C \text{ ....(2)},$$

then

$$\frac{d^2(r^2)}{dt^2} = 2C,$$

therefore $$r^2 = A + 2Bt + Ct^2 \text{ ....(3)}.$$

From (2) we find by differentiation

$$3P + r\frac{dP}{dr} = 0,$$

therefore $$P \propto \frac{1}{r^3}.$$

Hence the case in question is that of the inverse third power. It may be worth while to find θ in terms of t, and to obtain, by elimination of t, the equations of the orbits which are possible with such a force.

We have, in all central orbits,

$$r^2\frac{d\theta}{dt} = h \text{ ....(4)}.$$

Hence, in the present case, by (3),

$$\frac{d\theta}{dt} = \frac{h}{A + 2Bt + Ct^2} = \frac{h}{C}\cdot\frac{1}{\left(t + \frac{B}{C}\right)^2 + \frac{AC - B^2}{C^2}} \text{ ...(5)}.$$

Put now $$\tau = t + \frac{B}{C},$$

and we get $$r^2 = C\left\{\tau^2 + \frac{AC - B^2}{C^2}\right\} \quad\text{.................(3')},$$

and $$\frac{d\theta}{d\tau} = \frac{h}{C}\frac{1}{\tau^2 + \dfrac{AC - B^2}{C^2}} \quad\text{.................(5')}.$$

There are, of course, four cases.

I. $AC = B^2$. The integral of (5′) is

$$\theta + \alpha = -\frac{h}{C}\frac{1}{\tau};$$

and $$r = \pm\sqrt{C}\tau.$$

Here C must be *positive*. Hence

$$r = \mp\frac{h}{\sqrt{C}(\theta + \alpha)},$$

the equation of the reciprocal spiral.

II. $\dfrac{AC - B^2}{C^2} = a^2$. (3′) and (5′) give

$$\frac{aC}{h}(\theta + \alpha) = \tan^{-1}\frac{\tau}{a},$$

and therefore $$r^2 = Ca^2 \sec^2 \frac{aC}{h}(\theta + \alpha),$$

or $$r\cos\frac{aC}{h}(\theta + \alpha) = \sqrt{\frac{AC - B^2}{C}}.$$

III. $\dfrac{AC - B^2}{C^2} = -a^2$. Here

$$\frac{C}{h}(\theta + \alpha) = \frac{1}{2a}\log\frac{\tau - a}{\tau + a},$$

and $$r^2 = C(\tau^2 - a^2),$$

whence, after reduction,

$$r = \frac{2a\sqrt{C}}{\epsilon^{\frac{aC}{h}(\theta+\alpha)} - \epsilon^{-\frac{aC}{h}(\theta+\alpha)}} = \frac{1}{M\epsilon^{\frac{aC}{h}\theta} - N\epsilon^{-\frac{aC}{h}\theta}}.$$

IV. $C=0$.

$$\theta=\frac{h}{B}\log\frac{r}{M}, \text{ or } r=M\epsilon^{\frac{B}{h}\theta}.$$

These are, of course, the results of the integration of the usual equation between u and θ. [Compare Chap. V. Ex. (9).]

As another case, suppose in (1)

$$-2\int P dr - Pr = mr^2 + \frac{C}{2} \text{..................} (6).$$

Differentiate, multiply by r^2, and integrate, then

$$P = -\tfrac{1}{2}mr + \frac{n}{r^3}.$$

Hence, in the case of the direct first power, or a combination of this with the inverse third,

$$\frac{d^2(r^2)}{dt^2} = 2mr^2 + C,$$

which gives, according as m is positive or negative,

$$r^2 + \frac{C}{2m} = \begin{Bmatrix} M\epsilon^{t\sqrt{2m}} + N\epsilon^{-t\sqrt{2m}} \\ M\cos(t\sqrt{-2m}+N) \end{Bmatrix}.$$

By means of (4), these equations give us θ in terms of t, and, the latter being eliminated, we have the required orbit, which becomes the ellipse or hyperbola as usual when $n=0$, it being observed that we have an additional disposable constant introduced by the method employed in obtaining equation (1). It is evident that results of this kind may be multiplied indefinitely. To classify the cases in which the equations for r^2 and θ in terms of t can be completely integrated would be an interesting, but by no means an easy problem.

The method here employed is interesting as being that which is at once suggested by the application of Quaternions to the problem of Central Orbits. (Tait's *Quaternions*, § 345.)

As an additional example, take the gravitation case—then we obtain, as above,

$$\frac{d}{dt}\left(r\frac{dr}{dt}\right) = C + \frac{\mu}{r},$$

or
$$r\frac{dr}{dt} = \sqrt{C' + Cr^2 + 2\mu r}.$$

But $\quad v^2 = C + \dfrac{2\mu}{r}$. Hence, in ellipse,

$$C = -\frac{\mu}{a}.$$

Also $\quad \dfrac{dr}{dt} = 0$ for $r = a(1 \pm e)$. Thus

$$r\frac{dr}{dt} = \sqrt{\frac{\mu}{a}}\sqrt{a^2e^2 - (r-a)^2}.$$

The form of this suggests the assumption

$$r - a = -ae\cos u,$$

so that
$$a^2e(1 - e\cos u)\frac{du}{dt} = \sqrt{\frac{\mu}{a}}\,ae,$$

whence, as usual,

$$nt = u - e\sin u,$$

as in § (160) above.

Another mode of looking at this question is as follows:—Eliminate θ between the equations of the central orbit

$$\ddot{r} - r\dot{\theta}^2 = P, \quad r^2\dot{\theta} = h,$$

and we have

$$\ddot{r} = P + \frac{h^2}{r^3},$$

from which the above results are obtained at once.

The investigation of any central orbit is thus immediately reduced to a case of rectilinear motion.

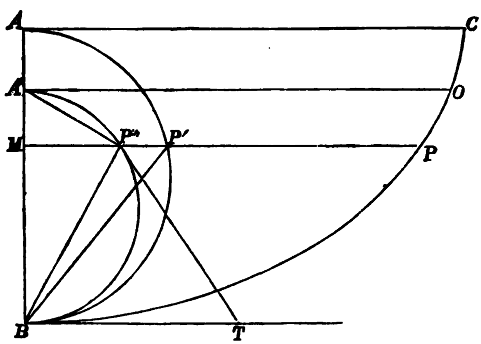
Another view of the same question, of which the above is only a special aspect, is Newton's *Revolving Orbit.* Suppose r to remain unaltered, as a function of the time, and θ to become $m\theta$—where m is constant. Then

$$\ddot{r} - r\dot{\theta}^2 = P + \frac{(m^2-1)\,h^2}{m^2 r^3}.$$

The central acceleration thus requires to be altered by a term depending on $\frac{1}{r^3}$ alone. This gives, by inspection, many of the results in Chap. V. above, e.g. Example 22, p. 155. [See on this subject, *Notes on Central Forces* by A. H. Curtis. *Messenger of Math.* April 1882.]

B. *To find the time of fall from rest down any arc of an inverted cycloid.*

Let O be the point from which the particle commences its motion. Draw OA' parallel to CA, and on BA' describe



a semicircle. Let P, P', P'' be corresponding points of the curve, the generating circle, and the circle just drawn, and let us compare the velocities of the particle at P, and the point P''. Let $P''T$ be the tangent at P''.

$$\frac{\text{velocity of } P''}{\text{velocity of } P} = \frac{\text{element at } P''}{\text{element at } P}$$

$$=\frac{P'T}{BP'}=\frac{P''T}{BP''}\sqrt{\frac{A'B}{AB}}$$

$$=\frac{A'B}{2A'P''}\sqrt{\frac{A'B}{AB}}.$$

But velocity of $P=\sqrt{(2g\,.\,A'M)}=\sqrt{\frac{2g}{A'B}}\,.\,A'P''$.

Hence velocity of $P''=\sqrt{\frac{g}{2AB}}\,.\,A'B$, a constant.

And, as the length of $A'P''B$ is $\frac{\pi}{2}\,.\,A'B$,

time from A' to B in circle = time from O to B in cycloid

$$=\pi\sqrt{\frac{AB}{2g}}.$$

COR. It is evident from the proof, that the particle descends half the vertical distance to B in half the time it takes to reach B.

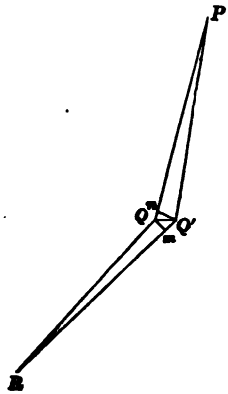
C. *To find the nature of the brachistochrone under gravity.*

The following is founded on Bernoulli's original solution. (WOODHOUSE, *Isoperimetrical Problems.*)

From Art. 180 it is evident that the curve lies in the vertical plane which contains the given points. Also it is easy to see that if the time of descent through the entire curve is a minimum, that through any portion of the curve is less than if that portion were changed into any other curve.

And it is obvious that, *between any two contiguous equal values of a continuously varying quantity, a maximum or minimum must lie.* [This principle, though excessively simple (witness its application to the barometer or thermometer), is of very great power, and often enables us to solve problems of maxima and minima, such as require in analysis not merely the processes of the Differential Calculus, but those of the Calculus of Variations. The present is a good example.]

Let, then, PQ, QR and PQ', $Q'R$ be two pairs of indefinitely small sides of polygons such that the time of de-



scending through either pair, starting from P with a given velocity, may be *equal*. Let QQ' be horizontal and indefinitely small compared with PQ and QR. The brachistochrone must lie *between* these paths, and must possess any property which they possess in common. Hence if v be the velocity down PQ (supposed uniform) and v' that down QR, drawing Qm, $Q'n$ perpendicular to RQ', PQ, we must have

$$\frac{Qn}{v} = \frac{Q'm}{v'}.$$

Now if θ be the inclination of PQ to the horizon, θ' that of QR, $Qn = QQ' \cos\theta$, $Q'm = QQ' \cos\theta'$. Hence the above equation becomes

$$\frac{\cos\theta}{v} = \frac{\cos\theta'}{v'}.$$

This is true for any two consecutive elements of the required curve; therefore throughout the curve

$$v \propto \cos\theta.$$

But $v^2 \propto$ vertical distance fallen through. (§ 173.) Hence the curve required is such that the cosine of the angle it makes with the horizontal line through the point of departure varies as the square root of the distance from that line; which is easily seen to be a property of the cycloid, if we remember that the tangent to that curve is parallel to the corresponding chord of its generating circle. For in the fig. p. 172,

$$\cos OPN = \cos OAP' = \frac{AP'}{AO} = \sqrt{\frac{AN}{AO}} \propto \sqrt{AN}.$$

The brachistochrone then, under gravity, is an inverted cycloid whose cusp is at the point from which the particle descends.

C_1. Were there any number of impressed forces we might suppose their resultant constant in magnitude and direction for two successive elements. Then reasoning similar to that in § 180 would shew that the osculating plane of the brachistochrone always contains the resultant force. Again we should have as in last Article,

$$\frac{\cos\theta}{v} = \frac{\cos\theta'}{v'},$$

where θ is now the complement of the angle between the curve and the resultant of the impressed forces.

Let that resultant $= F$, and let the element $PQ = \delta s$, and $\theta' = \theta + \delta\theta$. Then since F is supposed the same at P and Q,

$$v'^2 - v^2 = 2F\delta s \sin\theta \text{ (by Chap. IV.)},$$

or $$v\delta v = F\delta s \sin\theta.$$

But $v \propto \cos\theta$; which gives

$$\frac{\delta v}{v} = -\frac{\sin\theta}{\cos\theta}\delta\theta.$$

Hence $$\frac{v^2}{\frac{\delta s}{\delta\theta}} = -F\cos\theta.$$

But in the limit $\frac{\delta s}{\delta \theta} = \rho$, the radius of absolute curvature at Q, and $F \cos \theta$ is the normal component of the impressed force. Hence we obtain the result of § 185 for the general brachistochrone.

C_2. Now for the unconstrained path from P to R we have $\int v ds$ a minimum. Hence in the same way as before, ϕ being the angle corresponding to θ, $v \cos \phi = v' \cos \phi'$ from element to element, and therefore throughout the curve, if the *direction* of the force be constant.

But in the brachistochrone,

$$\frac{\cos \theta}{v} = \frac{\cos \theta'}{v'}.$$

Now if the velocities in the two paths be equal at any equipotential surface, they will be equal at every other. Hence taking the angles for any equipotential surface

$$\cos \theta \cos \phi = \text{constant}.$$

As an example, suppose a parabola with its vertex upwards to have for directrix the base of an inverted cycloid; these curves evidently satisfy the above condition, the one being the free path, the other the brachistochrone, for gravity, and the velocities being in each due to the same horizontal line. And it is seen at once that the product of the cosines of the angles which they make with any horizontal straight line which cuts both is a constant whose magnitude depends on that of the cycloid and parabola, its value being $\sqrt{\frac{l}{4a}}$ where l is the latus rectum of the parabola, and a the diameter of the generating circle of the cycloid.

D. *To shew that of two curves both concave in the sense of gravity, joining the same points in a vertical plane and not meeting in any other point, a particle will descend the enveloped in less time than it will the enveloping curve; the initial velocity being the same in both cases.*

Take the axis of x as the line to the level of which the initial velocity is due, and the axis of y in the direction of gravity, then

$$\frac{ds}{dt} = \sqrt{(2gy)};$$

$$\therefore t_1 = \int_{x_1}^{x_2} \frac{\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx}{\sqrt{(2gy)}}$$

$$\propto \int_{x_1}^{x_2} \frac{\sqrt{(1+p^2)}}{\sqrt{y}}\, dx \ldots\ldots\ldots\ldots\ldots\ldots(1);$$

$$\therefore \delta t_1 \propto \int_{x_1}^{x_2} dx\,\delta y \left(N - \frac{dP}{dx}\right),$$

(since the limits are constant),

$$\propto \int_{x_1}^{x_2} \delta y\, dx \left[-\tfrac{1}{2}\frac{\sqrt{(1+p^2)}}{y\sqrt{y}} + \tfrac{1}{2}\frac{p^2}{y\sqrt{y}\sqrt{(1+p^2)}} - \frac{q}{\sqrt{y}\sqrt{(1+p^2)}} + \frac{p^2 q}{\sqrt{y}(1+p^2)^{\frac{3}{2}}}\right]$$

$$\propto \int_{x_1}^{x_2} \delta y\, dx \frac{-(1+p^2) + 2p^2 qy - 2qy(1+p^2)}{2y\sqrt{y}(1+p^2)^{\frac{3}{2}}}$$

$$\propto \int_{x_1}^{x_2} \delta y\, dx \frac{-(1+p^2+2yq)}{2y\sqrt{y}(1+p^2)^{\frac{3}{2}}}.$$

Now the curve is convex to the axis of x, hence yq is positive, and by (1) $\sqrt{y}$ and $\sqrt{(1+p^2)}$ have the same sign. Hence the sign of δt_1 is the opposite of that of δy, and for an enveloping curve δy is negative. Hence the time of fall will be longer.

We may thus pass from one curve to any other enveloping one, even situated at a finite distance, provided the latter be concave throughout; else the multiplier of $\delta y \,.\, dx$ in the integral might change sign between the limits. (BERTRAND, *Liouville's Journal*, Vol. VII.)

A simple geometrical proof of this theorem may easily be obtained by drawing successive normals to the inner curve and producing them to meet the outer. The velocities in the pairs of arcs, thus cut out of the two curves, are equal (if the curves be indefinitely close), but the arcs themselves are generally longer in the outer curve, since the convexity of the inner curve is everywhere turned to it.

E. *To find the curve in which the time of descent to the lowest point is a given function* ϕ (a) *of* a *the vertical height fallen through.*

$$\text{Here } \sqrt{(2g)}\, t = \phi(a) = \int_0^a \frac{ds}{\sqrt{(a-x)}}.$$

Hence, the problem may be thus stated,

$$\textit{Having given} \qquad \phi(\mathrm{a}) = \int_0^{\mathrm{a}} \frac{\mathrm{ds}}{\sqrt{\mathrm{a}-\mathrm{x}}},$$

where ϕ *is a known function, find* s *in terms of* x. (ABEL, *Œuvres*, Tom. I.)

Put $ds = f'(x)\,dx$, divide by $\sqrt{z-a}$ and integrate both sides with regard to a, from $a=0$ to $a=z$.

$$\int_0^z \frac{\phi(a)\,da}{\sqrt{z-a}} = \int_0^z \frac{da}{\sqrt{z-a}} \int_0^a \frac{f'(x)\,dx}{\sqrt{a-x}}.$$

Changing the order of integration on the right-hand side, it becomes

$$\int_0^z \int_x^z \frac{f'(x)\,dx\,da}{\sqrt{(z-a)(a-x)}} = \pi\,\{f(z) - f(0)\}.$$

Hence

$$f(x) - f(0) = \frac{1}{\pi}\int_0^x \frac{\phi(a)\,da}{\sqrt{x-a}},$$

which is the required expression. (*Proc. R. S. E.*, 1874-5.)

Ex. I. Suppose the Tautochrone be required

$$\phi(a) = \sqrt{(2g)}\, t_0.$$

Here $$s = \frac{\sqrt{(2g)}\, t_0}{\pi} \int_0^x \frac{da}{(x-a)^{\frac{1}{2}}} = \frac{2\sqrt{(2g)}\, t_0}{\pi} x^{\frac{1}{2}};$$

or $$s^2 = \frac{8gt_0^2}{\pi^2} x;$$ the cycloid, as in § 175.

Ex. II. Let $\phi(a) = \sqrt{(2g)}\, \dfrac{a}{c}$, that is, let the time be proportional to the vertical height fallen through.

Here $$\frac{\pi cs}{\sqrt{(2g)}} = \int_0^x \frac{a\,da}{(x-a)^{\frac{1}{2}}} = x^{\frac{3}{2}} \frac{\Gamma(2)\,\Gamma(\frac{1}{2})}{\Gamma(2+\frac{1}{2})} = \frac{4}{3} x^{\frac{3}{2}},$$ the equation of the required curve.

THE END.

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